

2D Stochastic Chemotaxis-Navier-Stokes System

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February 16, 2017

Abstract: In this paper, we establish the existence and uniqueness of both mild(/variational) solutions and weak (in the sense of PDE) solutions of coupled system of 2D stochastic Chemotaxis-Navier-Stokes equations. The mild/variational solution is obtained through a fixed point argument in a purposely constructed Banach space. To get the weak solution we first prove the existence of a martingale weak solution and then we show that the pathwise uniqueness holds for the martingale solution.

Mathematics Subject Classification (2000). Primary: 60H15. Secondary: 35K55, 35K20

Key Words: Stochastic Chemotaxis-Navier-Stokes equations; Mild/variational solutions; Weak solutions; Energy estimates; Skorohod representation; Tightness; Pathwise uniqueness.

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1 Introduction

The purpose of this paper is to establish the existence and uniqueness of the solution of the coupled 2D stochastic Chemotaxis-Navier-Stokes system:

$$\begin{aligned} dn + u \cdot \nabla n dt &= \delta \Delta n dt - \nabla \cdot (\chi(c)n \nabla c) dt, \\ dc + u \cdot \nabla c dt &= \mu \Delta c dt - k(c)n dt, \\ du + (u \cdot \nabla) u dt + \nabla P dt &= \nu \Delta u dt - n \nabla \phi dt + \sigma(u) dW_t, \\ \nabla \cdot u &= 0, \quad t > 0, \quad x \in \mathcal{O}. \end{aligned} \tag{1.1}$$

The system arises in the modeling of bacterial suspensions in fluid drops and describes the spontaneous emergence of patterns in populations of oxygen-driven swimming bacteria. Here, $\mathcal{O} \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary $\partial\mathcal{O}$, which will be the spatial domain where the moving cells and the fluid interact. The unknowns are $n = n(t, x) : \mathbb{R}^+ \times \mathcal{O} \rightarrow \mathbb{R}^+$, $c = c(t, x) : \mathbb{R}^+ \times \mathcal{O} \rightarrow \mathbb{R}^+$, $u = u(t, x) : \mathbb{R}^+ \times \mathcal{O} \rightarrow \mathbb{R}^2$ and $P = P(t, x) : \mathbb{R}^+ \times \mathcal{O} \rightarrow \mathbb{R}$, which represent respectively the cell density, chemical concentration, velocity field and pressure of the fluid. Positive constants δ, μ, ν are the corresponding diffusion coefficients for the cells, chemical and fluid. The gravitational potential $\phi = \phi(x)$, the chemotactic sensitivity $\chi(c)$ and the per-capita oxygen consumption rate $k(c)$ are supposed to be given sufficiently smooth functions. $\{W_t, t \geq 0\}$ is a cylindrical Wiener process representing the external random driving force.

System (1.1) is considered with the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \text{ and } u = 0 \text{ for } x \in \partial\mathcal{O} \text{ and } t > 0, \tag{1.2}$$

and the initial conditions

$$n(0, x) = n_0(x), \quad c(0, x) = c_0(x), \quad u(0, x) = u_0(x), \quad x \in \mathcal{O}. \tag{1.3}$$

The deterministic models of system (1.1) (i.e. $\sigma = 0$) was proposed by Tuval *et al.* in [17]. In [25], the authors suggest a wider variants to describe more complicated interaction neighborhood environment around cells. The well-posedness of the deterministic models of system (1.1) (and its variants) is a highly non-trivial problem. In the past several years, the main focus of the existing literature is on the solvability of the system, see [2, 3, 5, 7, 9, 10, 11, 14, 20, 21, 22, 23, 27] and reference therein. We like to mention a few of them which are relevant to our work. In [12], local (in time) weak solutions (in the sense of PDE) were constructed in a bounded domain in \mathbb{R}^d , $d = 2, 3$ with no-flux boundary condition and in \mathbb{R}^2 for a special case. Based on some nice energy estimates, if the convective term $(u \cdot \nabla)u$ is neglected, global weak solutions were obtained in [4] provided the initial data or $\nabla\phi$ is small. Our work is motivated and influenced by the recent papers [11] and [20]. In [11], for the models in \mathbb{R}^2 , Liu and Lorz developed some nice entropy estimates to prove the global existence of weak solutions to the deterministic models of system (1.1) for large initial data. In [20], when $\mathcal{O} \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary $\partial\mathcal{O}$, the author managed to establish the existence and uniqueness of global strong (in the sense of PDE) solution of system (1.1) without the restriction of the smallness of either the initial data or the coefficients. There are many other interesting results on this topic, we refer to the references mentioned above. Finally, we refer the reader to [19, 24] for the stabilization and convergence rate of solutions of the deterministic models of system (1.1) and its variants.

Taking into account the random environment the bacteria are in and the effect of random external forces, it is natural to consider the coupled 2D stochastic Chemotaxis-Navier-Stokes system (1.1). Adding the singular random noise to the system changes the mathematical analysis significantly. In this paper we seek for probabilistically the so called pathwise/strong solutions. While in sense of PDE, we consider both the mild/variational solutions and the weak solutions under two different sets of conditions. From now on, the term of weak solutions are reserved for the weak solutions in the sense of PDE. The paper is divided into two parts. In the first part, we establish the existence and uniqueness of mild/variational solutions to system (1.1). To this end, we first appropriately cut off the coefficients of the system and construct a local (in time) mild/variational solution using fixed point arguments in a certain Banach space and we then show that the mild/variational solution is global by providing some energy estimates. In the second part, we obtain the existence and uniqueness of pathwise weak solution of the system (1.1). For this purpose, we first establish the existence of a martingale weak solution. In order to do so, we define a sequence of approximating systems and prove that a subsequence of the approximate solutions converges in law to a martingale weak solution of system (1.1). Then we prove that the pathwise uniqueness of weak solutions

holds. As an application of Watanabe and Yamada Theorem we obtain both the pathwise existence and uniqueness of the weak solution. Because the proofs of the main results are involved, we will state the main results in next section and leave the details of the arguments in the rest of the paper.

The paper is organized as follows. In Section 2, we spread out the precise assumptions and the framework. We also state the main results. Section 3 consists of several subsections. It is devoted to establishing the existence and uniqueness of mild/variational solution. The entire Section 4 is to prove the existence and uniqueness of the pathwise weak solution.

2 Framework and Statement of the Main Results

Let $L^q(\mathcal{O})$ denote the L^q space with respect to the Lebesgue measure. $W^{k,q}(\mathcal{O})$ denotes the Sobolev space of functions whose distributional derivatives of order up to k belong to L^q . Let A be the realization of the Stokes operator $-\mathcal{P}\Delta$, where \mathcal{P} denotes the Helmholtz projection from $L^2(\mathcal{O})$ into the space $H = \{\varphi \in L^2(\mathcal{O}) | \nabla \cdot \varphi = 0\}$. In the sequel, $\left(e^{t\Delta}\right)_{t \geq 0}$, $\left(e^{-tA}\right)_{t \geq 0}$ will denote respectively the Neumann heat semigroup and the Stokes semigroup with Dirichlet boundary condition.

For simplicity, we set $H^k(\mathcal{O}) := W^{k,2}(\mathcal{O})$,

$$\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathcal{O})}, \quad \|\cdot\|_{L^q} := \|\cdot\|_{L^q(\mathcal{O})}, \quad \|\cdot\|_\alpha := \|\cdot\|_{D(A^\alpha)}, \quad \|\cdot\|_{k,q} := \|\cdot\|_{W^{k,q}(\mathcal{O})}, \quad \|\cdot\|_{H^k} := \|\cdot\|_{W^{k,2}(\mathcal{O})}.$$

We introduce the following conditions on the parameters and functions involved in the system (1.1):

- (H.1) (a) $\chi \in C^2([0, \infty))$, $\chi > 0$ in $[0, \infty)$,
(b) $k \in C^2([0, \infty))$, $k(0) = 0$, $k > 0$ in $(0, \infty)$,
(c) $\phi \in C^2(\bar{\mathcal{O}})$,
(H.2) $\left(\frac{k(c)}{\chi(c)}\right)' > 0$, $\left(\frac{k(c)}{\chi(c)}\right)'' \leq 0$, $(\chi(c) \cdot k(c))' \geq 0$ on $[0, \infty)$.

Let U be a real Hilbert space and $\{W_t, t \geq 0\}$ a U -cylindrical Wiener process on a given complete, filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \geq 0, \mathbb{P})$, representing the driving external random force.

Let $\mathcal{L}^2(U, D(A^\beta))$ denote the space of Hilbert-Schmidt operators g from U into $D(A^\beta)$ and its norm is denoted by $\|g\|_{\mathcal{L}_\beta^2}$. For a mapping $\sigma : D(A^\beta) \rightarrow \mathcal{L}^2(U, D(A^\beta))$, we introduce the following hypothesis:

- (H.3) there exists a positive constant K such that for all $u_1, u_2, u \in H$,

$$\|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_0^2}^2 \leq K\|u_1 - u_2\|_H^2, \text{ and } \|\sigma(u)\|_{\mathcal{L}_0^2}^2 \leq K(1 + \|u\|_H^2),$$

where $\mathcal{L}_0^2 = \mathcal{L}^2(U, H)$,

- (H.4) there exists a positive constant K such that for all $u_1, u_2, u \in D(A^\alpha)$,

$$\|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_\alpha^2}^2 \leq K\|u_1 - u_2\|_\alpha^2, \text{ and } \|\sigma(u)\|_{\mathcal{L}_\alpha^2}^2 \leq K(1 + \|u\|_\alpha^2),$$

- (H.5) there exists a positive constant K such that for all $u_1, u_2, u \in D(A^{\frac{1}{2}})$,

$$\|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_{\frac{1}{2}}^2}^2 \leq K\|u_1 - u_2\|_{\frac{1}{2}}^2, \text{ and } \|\sigma(u)\|_{\mathcal{L}_{\frac{1}{2}}^2}^2 \leq K(1 + \|u\|_{\frac{1}{2}}^2).$$

Set: $u(t) = u(t, \cdot)$, $n(t) = n(t, \cdot)$ and $c(t) = c(t, \cdot)$. Let $q > 2$.

Definition 2.1 We say that (n, c, u) is a mild solution of system (1.1) if (n, c, u) is a progressively measurable stochastic processes with values in $C^0(\bar{\mathcal{O}}) \times W^{1,q}(\mathcal{O}) \times D(A^\alpha)$, which satisfies

$$\begin{aligned} n(t) &= e^{t\delta\Delta} n_0 - \int_0^t e^{(t-s)\delta\Delta} \left\{ u(s) \cdot \nabla n(s) \right\} ds - \int_0^t e^{(t-s)\delta\Delta} \left\{ \nabla \cdot \left(\chi(c(s)) n(s) \nabla c(s) \right) \right\} ds, \\ c(t) &= e^{t\mu\Delta} c_0 - \int_0^t e^{(t-s)\mu\Delta} \left\{ u(s) \cdot \nabla c(s) \right\} ds - \int_0^t e^{(t-s)\mu\Delta} \left\{ k(c(s)) n(s) \right\} ds, \\ u(t) &= e^{-t\nu A} u_0 - \int_0^t e^{-(t-s)\nu A} \mathcal{P} \left\{ (u(s) \cdot \nabla) u(s) \right\} ds - \int_0^t e^{-(t-s)\nu A} \mathcal{P} \left\{ n(s) \nabla \phi \right\} ds \\ &\quad + \int_0^t e^{-(t-s)\nu A} \sigma(u(s)) dW_s, \end{aligned} \tag{2.1}$$

P -a.s.

Remark 2.1 Note that $u(s) \cdot \nabla n(s) = \nabla \cdot (u(s)n(s))$ because $\nabla \cdot u(s) = 0$. Under the setting in the above definition, actually $n(\cdot) \in L^2_{loc}([0, \infty), W^{1,2}(\mathcal{O}))$ P -a.s., and (n, c, u) is equivalent to a variational solution of the system in the Gelfand triple $W^{1,2}(\mathcal{O}) \subset L^2(\mathcal{O}) \subset W^{1,2}(\mathcal{O})^*$, that is, (n, c, u) satisfies

$$\begin{aligned} n(t) + \int_0^t u(s) \cdot \nabla n(s) ds &= n_0 + \delta \int_0^t \Delta n(s) ds - \int_0^t \nabla \cdot (\chi(c(s))n(s)\nabla c(s)) ds, \\ c(t) + \int_0^t u(s) \cdot \nabla c(s) ds &= c_0 + \mu \int_0^t \Delta c(s) ds - \int_0^t k(c(s))n(s) ds, \\ u(t) + \int_0^t \mathcal{P}\{(u(s) \cdot \nabla)u(s)\} ds &= u_0 - \nu \int_0^t Au(s) ds - \int_0^t \mathcal{P}\{(n(s)\nabla\phi)\} ds \end{aligned} \quad (2.2)$$

$$+ \int_0^t \sigma(u(s)) dW_s, \quad (2.3)$$

P -a.s..

Here is our first main result.

Theorem 2.1 Assume

$$\begin{aligned} n_0 &\in C^0(\bar{\mathcal{O}}), \quad n_0 > 0 \text{ in } \bar{\mathcal{O}}, \\ c_0 &\in W^{1,q}(\mathcal{O}), \text{ for some } q > 2, \quad c_0 > 0 \text{ in } \bar{\mathcal{O}}, \\ u_0 &\in D(A^\alpha), \text{ for some } \alpha \in (1/2, 1), \end{aligned} \quad (2.4)$$

and the assumptions **(H.1)**-(**H.5**) hold. Then there exists a unique mild/variational solution to the system (1.1).

Define $V := D(A^{1/2})$ and its norm

$$\|u\|_V := \|A^{1/2}u\|_H = \|\nabla u\|_{L^2}.$$

Its dual space will be denoted by V^* .

Introduce the following conditions:

- (A) (a) $\chi(\cdot)$ and $k(\cdot)$ are smooth with $k(0) = 0$, $k(c) > 0$ in $(0, \infty)$ and $k'(c) \geq 0$, $\chi(c) > 0$ for every $c \in \mathbb{R}$,
- (b) $\chi'(c) \geq 0$, $(\frac{k(c)}{\chi(c)})' > 0$, $(\frac{k(c)}{\chi(c)})'' < 0$, $(\chi(c) \cdot k(c))' > 0$ on $[0, \infty)$,
- (c) $\phi \in C^2(\bar{\mathcal{O}})$,
- (B) (n_0, c_0, u_0) satisfies
 - (B1) $n_0(x) \geq 0$, $0 \leq c_0(x) \leq C_M < \infty$, $\nabla \cdot u_0(x) = 0$ on $x \in \mathcal{O}$,
 - (B2) $u_0 \in H$,
 - (B3) $n_0(1 + |x| + |\ln n_0|) \in L^1(\mathcal{O})$,
 - (B4) $\nabla c_0 \in L^2(\mathcal{O})$, $\nabla \Psi(c_0) \in L^2(\mathcal{O})$ where

$$\Psi(c) = \int_0^c \sqrt{\frac{\chi(s)}{k(s)}} ds.$$

- (C) for any $u, u_1, u_2 \in H$,

$$\|\sigma(u)\|_{\mathcal{L}_0^2}^2 \leq C(1 + \|u\|_H^2) \text{ and } \|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_0^2}^2 \leq C\|u_1 - u_2\|_H^2.$$

Definition 2.2 We say that (n, c, u) is a weak solution to the system (1.1) if (n, c, u) is a progressively measurable process that satisfies, for any $T > 0$,

- (1) P -a.s.

$$\begin{aligned} n(1 + |x| + |\ln n|) &\in L^\infty([0, T], L^1(\mathcal{O})), \quad \nabla \sqrt{n} \in L^2([0, T], L^2(\mathcal{O})), \\ c &\in L^\infty([0, T], L^\infty(\mathcal{O}) \cap H^1(\mathcal{O})) \cap L^2([0, T], H^2(\mathcal{O})), \\ u &\in C([0, T], H) \cap L^2([0, T], V); \end{aligned}$$

(2) For all $\psi_1, \psi_2 \in C^\infty([0, T] \times \mathcal{O})$ with compact supports in the space variable, and $\psi_1(T, \cdot) = \psi_2(T, \cdot) = 0$, P -a.s.

$$\int_{\mathcal{O}} \psi_1(0, x) n_0 dx = \int_0^T \int_{\mathcal{O}} n [\partial_t \psi_1 + \nabla \psi_1 \cdot u + \delta \Delta \psi_1 + \nabla \psi_1 \cdot (\chi(c) \nabla c)] dx dt,$$

$$\int_{\mathcal{O}} \psi_2(0, x) c_0 dx = \int_0^T \int_{\mathcal{O}} c [\partial_t \psi_2 + \nabla \psi_2 \cdot u + \mu \Delta \psi_2] - nk(c) \psi_2 dx dt,$$

(3) For all $e \in V$, $0 \leq t \leq T$,

$$\begin{aligned} \langle u(t), e \rangle_{H,H} &= \langle u_0, e \rangle_{H,H} - \int_0^t \nu \langle Au(s), e \rangle_{V^*, V} ds - \int_0^t \langle (u(s) \cdot \nabla) u(s), e \rangle_{V^*, V} ds \\ &\quad - \int_0^t \langle n(s) \nabla \phi, e \rangle_{H,H} ds + \int_0^t \langle \sigma(u(s)) dW_s, e \rangle_{H,H} ds \end{aligned}$$

holds P -a.s.

The following is our second main result.

Theorem 2.2 Assume the assumptions (A)-(C) hold, and the function $\chi(\cdot)$ is a positive constant. Then there exists a unique weak solution to the system (1.1).

We end this section by recalling the following two properties of the solution (see Lemma 2.2 in [20]). The first property follows by integrating the first equation in the system (1.1). The second one is a consequence of the comparison theorem/maximum principle.

Lemma 2.1 The solution of (1.1) satisfies, for all $t \geq 0$,

$$\int_{\mathcal{O}} n(t, x) dx = \int_{\mathcal{O}} n_0(x) dx, \tag{2.5}$$

and

$$\|c(t, \cdot)\|_\infty \leq \|c_0\|_\infty, \quad n(t, x) \geq 0, \quad c(t, x) \geq 0. \tag{2.6}$$

Using (2.5) and the Gagliardo-Nirenberg-Sobolev inequality, we also have

$$\begin{aligned} \|n(t)\|_{L^2} &\leq C \left(\|n(t)\|_{L^1}^{1/2} \|\nabla \sqrt{n(t)}\|_{L^2} + \|\sqrt{n(t)}\|_{L^2}^2 \right) \\ &\leq C \left(\|n_0\|_{L^1}^{1/2} \|\nabla \sqrt{n(t)}\|_{L^2} + \|n_0\|_{L^1} \right). \end{aligned} \tag{2.7}$$

3 Existence and Uniqueness of Mild/Variational Solutions

In this section, we assume that conditions (H.1)-(H.5) hold. Our aim is to prove Theorem 2.1.

3.1 Existence of Local Solutions

Introduce the following spaces

$$\Upsilon_t^n := L^\infty([0, t], C^0(\bar{\mathcal{O}})), \quad \Upsilon_t^c := L^\infty([0, t], W^{1,q}(\mathcal{O})), \quad \Upsilon_t^u := L^\infty([0, t], D(A^\alpha))$$

with the corresponding norms given by

$$\|n\|_{\Upsilon_t^n} = \sup_{s \in [0, t]} \|n(s)\|_\infty, \quad \|c\|_{\Upsilon_t^c} = \sup_{s \in [0, t]} \|c(s)\|_{1,q}, \quad \|u\|_{\Upsilon_t^u} = \sup_{s \in [0, t]} \|u(s)\|_\alpha.$$

Definition 3.1 We say that (n, c, u, τ) is a local mild/variational solution of system (1.1) if

(1) τ is a stopping time and (n, c, u) is a progressively measurable stochastic processes with values in $C^0(\bar{\mathcal{O}}) \times W^{1,q}(\mathcal{O}) \times D(A^\alpha)$,

(2) there exists a nondecreasing sequence of stopping times $\{\tau_l, l \geq 1\}$ with $\tau_l \uparrow \tau$ a.s. as $l \uparrow \infty$, such that $\{(n(t \wedge \tau_l), c(t \wedge \tau_l), u(t \wedge \tau_l)), t \geq 0\}$ is a mild/variational solution to system (1.1).

Theorem 3.1 There exists a local mild/variational solution to the system (1.1).

Proof. To use a cut off argument, we will modify the coefficients in system (1.1). Fix a function $\theta \in C^2([0, \infty), [0, 1])$ such that

- (1) $\theta(r) = 1, r \in [0, 1]$,
- (2) $\theta(r) = 0, r > 2$,
- (3) $\sup_{r \in [0, \infty)} |\theta'(r)| \leq C < \infty$.

Set $\theta_m(\cdot) = \theta(\frac{\cdot}{m})$. For every $m \geq 1$, consider the following system of SPDEs

$$\begin{aligned} dn + \theta_m(\|u\|_{\Upsilon_t^u})\theta_m(\|n\|_{\Upsilon_t^n})u \cdot \nabla n dt &= \delta \Delta n dt - \theta_m(\|n\|_{\Upsilon_t^n})\theta_m(\|c\|_{\Upsilon_t^c})\nabla \cdot (\chi(c)n \nabla c) dt, \\ dc + \theta_m(\|u\|_{\Upsilon_t^u})\theta_m(\|c\|_{\Upsilon_t^c})u \cdot \nabla c dt &= \mu \Delta c dt - \theta_m(\|c\|_{\Upsilon_t^c})\theta_m(\|n\|_{\Upsilon_t^n})k(c)n dt, \\ du + \theta_m(\|u\|_{\Upsilon_t^u})(u \cdot \nabla)u dt + \nabla P dt &= \nu \Delta u dt - \theta_m(\|n\|_{\Upsilon_t^n})n \nabla \phi dt + \sigma(u) dW_t, \\ \nabla \cdot u &= 0, \quad t > 0, \quad x \in \mathcal{O}. \end{aligned} \quad (3.1)$$

To simplify the exposition, we assume $\delta = \mu = \nu = 1$, $\chi(c) = 1$, and $k(c) = c$. The general case is entirely similar.

Let S_T be the space of all $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted, $C^0(\bar{\mathcal{O}}) \times W^{1,q}(\mathcal{O}) \times D(A^\alpha)$ -valued stochastic processes $(n(t), c(t), u(t)), t \geq 0$ such that

$$\|(n, c, u)\|_{S_T}^2 := \mathbb{E}\left(\|n\|_{\Upsilon_T^n}^2\right) + \mathbb{E}\left(\|c\|_{\Upsilon_T^c}^2\right) + \mathbb{E}\left(\|u\|_{\Upsilon_T^u}^2\right) < \infty.$$

Then S_T equipped with the norm $\|\cdot\|_{S_T}$ is a Banach space.

We introduce a mapping $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ on S_T by defining

$$\begin{aligned} \Phi_1(n, c, u)(t) &:= e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta} \left\{ \theta_m(\|n\|_{\Upsilon_s^n})\theta_m(\|c\|_{\Upsilon_s^c})\nabla \cdot (n \nabla c) \right. \\ &\quad \left. + \theta_m(\|u\|_{\Upsilon_s^u})\theta_m(\|n\|_{\Upsilon_s^n})\nabla \cdot (un) \right\}(s) ds, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \Phi_2(n, c, u)(t) &:= e^{t\Delta}c_0 - \int_0^t e^{(t-s)\Delta} \left\{ \theta_m(\|n\|_{\Upsilon_s^n})\theta_m(\|c\|_{\Upsilon_s^c})nc \right. \\ &\quad \left. + \theta_m(\|u\|_{\Upsilon_s^u})\theta_m(\|c\|_{\Upsilon_s^c})u \cdot \nabla c \right\}(s) ds, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \Phi_3(n, c, u)(t) &:= e^{-tA}u_0 - \int_0^t e^{-(t-s)A} \theta_m(\|u\|_{\Upsilon_s^u}) \mathcal{P}\{(u(s) \cdot \nabla)u(s)\} ds \\ &\quad - \int_0^t e^{-(t-s)A} \theta_m(\|n\|_{\Upsilon_s^n}) \mathcal{P}\{n(s) \nabla \phi\} ds + \int_0^t e^{-(t-s)A} \sigma(u(s)) dW_s. \end{aligned} \quad (3.4)$$

Let B denote the operator $-\Delta + 1$ in $L^q(\mathcal{O})$ ($q > 2$) equipped with Neumann boundary condition. Then, for $\beta \in (\frac{1}{q}, \frac{1}{2})$, we have the continuous imbedding $D(B^\beta) \hookrightarrow C^0(\bar{\mathcal{O}})$. Using a similar argument as that in

[20] (page 325), we have

$$\begin{aligned}
& \|\Phi_1(n, c, u)(t)\|_\infty \\
& \leq \|e^{t\Delta}n_0\|_\infty + \varrho \int_0^t \|B^\beta e^{-(t-s)(B-1)} \theta_m(\|n\|_{\Upsilon_s^n}) \theta_m(\|c\|_{\Upsilon_s^c}) \nabla \cdot (n \nabla c)\|_{L^q} ds \\
& \quad + \varrho \int_0^t \|B^\beta e^{-(t-s)(B-1)} \left(\theta_m(\|u\|_{\Upsilon_s^u}) \theta_m(\|n\|_{\Upsilon_s^n}) \nabla \cdot (un) \right)\|_{L^q} ds \\
& \leq \|n_0\|_\infty + \varrho \int_0^t (t-s)^{-\beta-\frac{1}{2}} \theta_m(\|n\|_{\Upsilon_s^n}) \theta_m(\|c\|_{\Upsilon_s^c}) \|n \nabla c\|_{L^q} ds \\
& \quad + \varrho \int_0^t (t-s)^{-\beta-\frac{1}{2}} \theta_m(\|u\|_{\Upsilon_s^u}) \theta_m(\|n\|_{\Upsilon_s^n}) \|un\|_{L^q} ds \\
& \leq \|n_0\|_\infty + \varrho m^2 \int_0^t (t-s)^{-\beta-\frac{1}{2}} ds \\
& \leq \|n_0\|_\infty + \varrho m^2 T^{\frac{1}{2}-\beta}, \quad \forall t \in [0, T],
\end{aligned} \tag{3.5}$$

here we have used the continuous imbedding $D(A^\alpha) \hookrightarrow C^0(\bar{O})$.

Fix any $\gamma \in (\frac{1}{2}, 1)$,

$$\begin{aligned}
& \|\Phi_2(n, c, u)(t)\|_{1,q} \\
& \leq \|e^{t\Delta}c_0\|_{1,q} + \varrho \int_0^t \|B^\gamma e^{-(t-s)(B-1)} \theta_m(\|n\|_{\Upsilon_s^n}) \theta_m(\|c\|_{\Upsilon_s^c}) nc\|_{L^q} ds \\
& \quad + \varrho \int_0^t \|B^\gamma e^{-(t-s)(B-1)} \left(\theta_m(\|u\|_{\Upsilon_s^u}) \theta_m(\|c\|_{\Upsilon_s^c}) u \cdot \nabla c \right)\|_{L^q} ds \\
& \leq \varrho \|c_0\|_{1,q} + \varrho \int_0^t (t-s)^{-\gamma} \theta_m(\|n\|_{\Upsilon_s^n}) \theta_m(\|c\|_{\Upsilon_s^c}) \|nc\|_{L^q} ds \\
& \quad + \varrho \int_0^t (t-s)^{-\gamma} \theta_m(\|u\|_{\Upsilon_s^u}) \theta_m(\|c\|_{\Upsilon_s^c}) \|(u \cdot \nabla c)\|_{L^q} ds \\
& \leq \varrho \|c_0\|_{1,q} + \varrho m^2 \int_0^t (t-s)^{-\gamma} ds \\
& \leq \varrho \|c_0\|_{1,q} + \varrho m^2 T^{1-\gamma}, \quad \forall t \in [0, T].
\end{aligned} \tag{3.6}$$

For Φ_3 , we have

$$\begin{aligned}
& \|A^\alpha \Phi_3(n, c, u)(t)\|_{L^2} \\
& \leq \|e^{-tA} A^\alpha u_0\|_{L^2} + \int_0^t \|e^{-(t-s)A} A^\alpha \theta_m(\|u\|_{\Upsilon_s^u}) \mathcal{P}\{(u(s) \cdot \nabla)u(s)\}\|_{L^2} ds \\
& \quad + \int_0^t \|e^{-(t-s)A} A^\alpha \theta_m(\|n\|_{\Upsilon_s^n}) \mathcal{P}\{n(s) \nabla \phi\}\|_{L^2} ds + \left\| \int_0^t e^{-(t-s)A} A^\alpha \sigma(u(s)) dW_s \right\|_{L^2} \\
& \leq \|u_0\|_\alpha + I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{3.7}$$

Noticing

$$\|(u \cdot \nabla)u\|_{L^2} \leq \|u\|_\infty \|\nabla u\|_{L^2} \leq C \|A^\alpha u\|_{L^2}^2,$$

we have

$$I_1(t) \leq C \int_0^t (t-s)^{-\alpha} \theta_m(\|u\|_{\Upsilon_s^u}) \|A^\alpha u(s)\|_{L^2}^2 ds \leq C m^2 \int_0^t (t-s)^{-\alpha} ds \leq C m^2 t^{1-\alpha}. \tag{3.8}$$

For I_2 , we have

$$I_2(t) \leq \int_0^t (t-s)^{-\alpha} \theta_m(\|n\|_{\Upsilon_s^n}) \|n(s) \nabla \phi\|_{L^2} ds \leq m \|\nabla \phi\|_\infty \int_0^t (t-s)^{-\alpha} ds \leq C m t^{1-\alpha}. \tag{3.9}$$

To estimate I_3 , let $Z(t) := \int_0^t e^{-(t-s)A} A^\alpha \sigma(u(s)) dW_s$. Then Z is the solution of the evolution equation

$$\begin{aligned}
dZ(t) &= -AZ(t)dt + A^\alpha \sigma(u(t))dW_t, \\
Z(0) &= 0.
\end{aligned}$$

Applying Itô's Formula, and then the BDG inequality, we have

$$\begin{aligned}
& \mathbb{E}(\sup_{t \in [0, T]} \|Z(t)\|_{L^2}^2) + 2E[\int_0^T \|Z(t)\|_{1/2}^2 dt] \\
& \leq 2\mathbb{E}\left(\sup_{t \in [0, T]} \left| \int_0^t \langle Z(s), A^\alpha \sigma(u(s)) dW_s \rangle_{L^2} \right| \right) + \mathbb{E}\left(\int_0^T \|A^\alpha \sigma(u(s))\|_{L_0^2}^2 ds\right) \\
& \leq 1/2\mathbb{E}(\sup_{t \in [0, T]} \|Z(t)\|_{L^2}^2) + C\mathbb{E}\left(\int_0^T \|A^\alpha \sigma(u(s))\|_{L_0^2}^2 ds\right) \\
& \leq 1/2\mathbb{E}(\sup_{t \in [0, T]} \|Z(t)\|_{L^2}^2) + C\mathbb{E}\left(\int_0^T 1 + \|A^\alpha u(s)\|_{L^2}^2 ds\right) \\
& \leq 1/2\mathbb{E}(\sup_{t \in [0, T]} \|Z(t)\|_{L^2}^2) + CT\left(1 + \mathbb{E}\left(\|u\|_{\Upsilon_T^u}^2\right)\right),
\end{aligned}$$

here we have used Assumption (H.4).

Hence

$$\mathbb{E}(\sup_{t \in [0, T]} \|I_3(t)\|_{L^2}^2) \leq CT\left(1 + \mathbb{E}(\|u\|_{\Upsilon_T^u}^2)\right). \quad (3.10)$$

Combining (3.7)–(3.10), we get

$$\mathbb{E}(\|\Phi_3(n, c, u)\|_{\Upsilon_T^u}^2) \leq C\left(\|u_0\|_\alpha^2 + m^4 T^{2-2\alpha} + T\mathbb{E}\left(\|u\|_{\Upsilon_T^u}^2\right) + T\right). \quad (3.11)$$

(3.5), (3.6) and (3.11) together show that Φ maps S_T into itself.

Next we will prove that if $T > 0$ is small enough then Φ can be made a contraction on S_T .

Let $(n_1, c_1, u_1), (n_2, c_2, u_2) \in S_T$. We have

$$\begin{aligned}
& \|\Phi_1(n_1, c_1, u_1)(t) - \Phi_1(n_2, c_2, u_2)(t)\|_\infty \\
& \leq \varrho \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot \left(\theta_m(\|n_1\|_{\Upsilon_s^n}) \theta_m(\|c_1\|_{\Upsilon_s^c}) n_1 \nabla c_1 - \theta_m(\|n_2\|_{\Upsilon_s^n}) \theta_m(\|c_2\|_{\Upsilon_s^c}) n_2 \nabla c_2 \right) ds \right\|_\infty \\
& \quad + \varrho \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot \left(\theta_m(\|u_1\|_{\Upsilon_s^u}) \theta_m(\|n_1\|_{\Upsilon_s^n}) u_1 n_1 - \theta_m(\|u_2\|_{\Upsilon_s^u}) \theta_m(\|n_2\|_{\Upsilon_s^n}) u_2 n_2 \right) ds \right\|_\infty \\
& := I_1(t) + I_2(t),
\end{aligned} \quad (3.12)$$

where ϱ is some generic constant. Similar to the proof of (3.5), we have

$$\begin{aligned}
& I_1(t) \\
& \leq \varrho \int_0^t (t-s)^{-\beta-\frac{1}{2}} \|\theta_m(\|n_1\|_{\Upsilon_s^n}) \theta_m(\|c_1\|_{\Upsilon_s^c}) n_1 \nabla c_1 - \theta_m(\|n_2\|_{\Upsilon_s^n}) \theta_m(\|c_2\|_{\Upsilon_s^c}) n_2 \nabla c_2\|_{L^q} ds.
\end{aligned} \quad (3.13)$$

Set

$$J(s) = \|\theta_m(\|n_1\|_{\Upsilon_s^n}) \theta_m(\|c_1\|_{\Upsilon_s^c}) n_1(s) \nabla c_1(s) - \theta_m(\|n_2\|_{\Upsilon_s^n}) \theta_m(\|c_2\|_{\Upsilon_s^c}) n_2(s) \nabla c_2(s)\|_{L^q}.$$

We will distinguish six cases to bound J . By the property of θ and the Minkowski inequality, we have the following estimates.

(J1) Suppose $\|n_1\|_{\Upsilon_s^n} \vee \|c_1\|_{\Upsilon_s^c} \vee \|n_2\|_{\Upsilon_s^n} \vee \|c_2\|_{\Upsilon_s^c} \leq 2m$. We have

$$\begin{aligned}
J(s) &= \|\theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_1\|_{\Upsilon_s^c})n_1\nabla c_1 - \theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c})n_2\nabla c_2\|_{L^q} \\
&\leq \|\theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_1\|_{\Upsilon_s^c})n_1\nabla c_1 - \theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c})n_1\nabla c_1\|_{L^q} \\
&\quad + \|\theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c})n_1\nabla c_1 - \theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c})n_2\nabla c_2\|_{L^q} \\
&\leq \left| \theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_1\|_{\Upsilon_s^c}) - \theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c}) \right| \|n_1(s)\nabla c_1(s)\|_{L^q} \\
&\quad + \|n_1\nabla c_1(s) - n_2\nabla c_2(s)\|_{L^q} \\
&\leq \left| \theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_1\|_{\Upsilon_s^c}) - \theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c}) \right| \|n_1(s)\|_{\infty} \|\nabla c_1(s)\|_{L^q} \\
&\quad + \|n_1\|_{\infty} \|c_1(s) - c_2(s)\|_{1,q} + \|n_1(s) - n_2(s)\|_{\infty} \|c_2(s)\|_{1,q} \\
&\leq 4m^2 \left(\left| \theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_1\|_{\Upsilon_s^c}) - \theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c}) \right| \right. \\
&\quad \left. + \left| \theta_m(\|n_1\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c}) - \theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c}) \right| \right) \\
&\quad + 2m(\|c_1(s) - c_2(s)\|_{1,q} + \|n_1(s) - n_2(s)\|_{\infty}) \\
&\leq 4m^2 \left(\left| \theta_m(\|c_1\|_{\Upsilon_s^c}) - \theta_m(\|c_2\|_{\Upsilon_s^c}) \right| + \left| \theta_m(\|n_1\|_{\Upsilon_s^n}) - \theta_m(\|n_2\|_{\Upsilon_s^n}) \right| \right) \\
&\quad + 2m(\|c_1(s) - c_2(s)\|_{1,q} + \|n_1(s) - n_2(s)\|_{\infty}) \\
&\leq 4m^2 \frac{C}{m} \left(\|c_1 - c_2\|_{\Upsilon_s^c} + \|n_1 - n_2\|_{\Upsilon_s^n} \right) + 2m(\|c_1(s) - c_2(s)\|_{1,q} + \|n_1(s) - n_2(s)\|_{\infty}) \\
&\leq Cm \left(\|c_1 - c_2\|_{\Upsilon_s^c} + \|n_1 - n_2\|_{\Upsilon_s^n} \right).
\end{aligned}$$

(J2) Suppose $\|n_1\|_{\Upsilon_s^n} \vee \|c_1\|_{\Upsilon_s^c} > 2m$ and $\|n_2\|_{\Upsilon_s^n} \vee \|c_2\|_{\Upsilon_s^c} > 2m$. We have

$$J(s) = 0.$$

(J3) Suppose $\|n_1\|_{\Upsilon_s^n} > 2m$ and $\|n_2\|_{\Upsilon_s^n} \vee \|c_2\|_{\Upsilon_s^c} \leq 2m$.

$$\begin{aligned}
J(s) &= \|\theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c})n_2\nabla c_2\|_{L^q} \\
&= |\theta_m(\|n_2\|_{\Upsilon_s^n}) - \theta_m(\|n_1\|_{\Upsilon_s^n})| \theta_m(\|c_2\|_{\Upsilon_s^c}) \|n_2\nabla c_2\|_{L^q} \\
&\leq Cm \|n_1 - n_2\|_{\Upsilon_s^n}.
\end{aligned}$$

(J4) Suppose $\|c_1\|_{\Upsilon_s^n} > 2m$ and $\|n_2\|_{\Upsilon_s^n} \vee \|c_2\|_{\Upsilon_s^c} \leq 2m$.

$$\begin{aligned}
J(s) &= \|\theta_m(\|n_2\|_{\Upsilon_s^n})\theta_m(\|c_2\|_{\Upsilon_s^c})n_2\nabla c_2\|_{L^q} \\
&= |\theta_m(\|c_2\|_{\Upsilon_s^c}) - \theta_m(\|c_1\|_{\Upsilon_s^c})| \theta_m(\|n_2\|_{\Upsilon_s^n}) \|n_2\nabla c_2\|_{L^q} \\
&\leq Cm \|c_1 - c_2\|_{\Upsilon_s^c}.
\end{aligned}$$

The proofs of the following two cases are similar as (J3) and (J4).

(J5) If $\|n_2\|_{\Upsilon_s^n} > 2m$ and $\|n_1\|_{\Upsilon_s^n} \vee \|c_1\|_{\Upsilon_s^c} \leq 2m$, then

$$J(s) \leq Cm \|n_1 - n_2\|_{\Upsilon_s^n}.$$

(J6) Suppose $\|c_2\|_{\Upsilon_s^n} > 2m$ and $\|n_1\|_{\Upsilon_s^n} \vee \|c_1\|_{\Upsilon_s^c} \leq 2m$. Then

$$J(s) \leq Cm \|c_1 - c_2\|_{\Upsilon_s^c}.$$

Putting (J1)–(J6) together, we get

$$J(s) \leq Cm (\|c_1 - c_2\|_{\Upsilon_s^c} + \|n_1 - n_2\|_{\Upsilon_s^n}). \quad (3.14)$$

Substituting (3.14) into (3.13), we get

$$\begin{aligned}
I_1(t) &\leq C \varrho m (\|c_1 - c_2\|_{\Upsilon_T^c} + \|n_1 - n_2\|_{\Upsilon_T^n}) \int_0^t (t-s)^{-\beta-\frac{1}{2}} ds \\
&\leq C \varrho m (\|c_1 - c_2\|_{\Upsilon_T^c} + \|n_1 - n_2\|_{\Upsilon_T^n}) T^{\frac{1}{2}-\beta}.
\end{aligned} \quad (3.15)$$

Using the similar arguments as in the proof of (3.14), we can show

$$\begin{aligned}
&\|\theta_m(\|u_1\|_{\Upsilon_s^u})\theta_m(\|n_1\|_{\Upsilon_s^n})u_1n_1 - \theta_m(\|u_2\|_{\Upsilon_s^u})\theta_m(\|n_2\|_{\Upsilon_s^n})u_2n_2\|_{L^q} \\
&\leq Cm (\|u_1 - u_2\|_{\Upsilon_s^u} + \|n_1 - n_2\|_{\Upsilon_s^n}).
\end{aligned}$$

Thus, similar to (3.15), we have

$$\begin{aligned} I_2(t) &\leq \varrho \int_0^t (t-s)^{-\beta-\frac{1}{2}} \|\theta_m(\|u_1\|_{\Upsilon_s^u})\theta_m(\|n_1\|_{\Upsilon_s^n})u_1n_1 - \theta_m(\|u_2\|_{\Upsilon_s^u})\theta_m(\|n_2\|_{\Upsilon_s^n})u_2n_2\|_{L^q} ds \\ &\leq C\varrho m(\|u_1 - u_2\|_{\Upsilon_T^u} + \|n_1 - n_2\|_{\Upsilon_T^n}) T^{\frac{1}{2}-\beta}. \end{aligned} \quad (3.16)$$

Substitute (3.15) and (3.16) into (3.12) to get

$$\begin{aligned} &\|\Phi_1(n_1, c_1, u_1)(t) - \Phi_1(n_2, c_2, u_2)(t)\|_\infty \\ &\leq C\varrho m\left(\|c_1 - c_2\|_{\Upsilon_T^c} + \|u_1 - u_2\|_{\Upsilon_T^u} + \|n_1 - n_2\|_{\Upsilon_T^n}\right) T^{\frac{1}{2}-\beta}, \end{aligned} \quad (3.17)$$

for $t \leq T$.

By a similar reasoning, we can show that

$$\begin{aligned} &\|\Phi_2(n_1, c_1, u_1)(t) - \Phi_2(n_2, c_2, u_2)(t)\|_{1,q} \\ &\leq C\varrho m\left(\|c_1 - c_2\|_{\Upsilon_T^c} + \|u_1 - u_2\|_{\Upsilon_T^u} + \|n_1 - n_2\|_{\Upsilon_T^n}\right) T^{1-\gamma}, \end{aligned} \quad (3.18)$$

for $t \leq T$, here γ is a number in $(\frac{1}{2}, 1)$.

Now we estimate $\|\Phi_3(n_1, c_1, u_1) - \Phi_3(n_2, c_2, u_2)\|_{\Upsilon_T^u}$. We have

$$\begin{aligned} &\|\Phi_3(n_1, c_1, u_1)(t) - \Phi_3(n_2, c_2, u_2)(t)\|_\alpha \\ &\leq \varrho \int_0^t (t-s)^{-\alpha} \|\theta_m(\|u_1\|_{\Upsilon_s^u})(u_1 \cdot \nabla)u_1 - \theta_m(\|u_2\|_{\Upsilon_s^u})(u_2 \cdot \nabla)u_2\|_{L^2} ds \\ &\quad + \varrho \int_0^t (t-s)^{-\alpha} \|\theta_m(\|n_1\|_{\Upsilon_s^n})n_1 \nabla \phi - \theta_m(\|n_2\|_{\Upsilon_s^n})n_2 \nabla \phi\|_{L^2} ds \\ &\quad + \varrho \left\| \int_0^t e^{-(t-s)A} A^\alpha (\sigma(u_1) - \sigma(u_2)) dW_s \right\|_{L^2} \\ &= \Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t). \end{aligned} \quad (3.19)$$

Using the similar arguments as in the proof of (3.15), it can be shown that

$$\Gamma_2(t) \leq C\varrho \|n_1 - n_2\|_{\Upsilon_T^n} \cdot T^{1-\alpha}, \quad (3.20)$$

for $t \leq T$. Note that $\Gamma_3(t) = \|Z(t)\|_{L^2}$, where Z satisfies the following SPDE

$$\begin{aligned} dZ(s) &= -AZ(s)ds + A^\alpha(\sigma(u_1) - \sigma(u_2))dW_s, \\ Z(0) &= 0. \end{aligned}$$

Using Itô's Formula and the BDG inequality, we have

$$\begin{aligned} &\mathbb{E}\left(\sup_{t \in [0, T]} \|Z(t)\|_{L^2}^2\right) + 2\mathbb{E}\left(\int_0^T \|Z(t)\|_{\frac{1}{2}}^2 dt\right) \\ &\leq \mathbb{E}\left(\int_0^T \|A^\alpha(\sigma(u_1(t)) - \sigma(u_2(t)))\|_{\mathcal{L}_0^2}^2 dt\right) \\ &\quad + 2\mathbb{E}\left(\sup_{t \in [0, T]} \left| \int_0^t \left\langle Z(s), A^\alpha(\sigma(u_1(s)) - \sigma(u_2(s)))dW_s \right\rangle \right|\right) \\ &\leq \varrho T \mathbb{E}\left(\|u_1 - u_2\|_{\Upsilon_T^u}^2\right) + \frac{1}{2}\mathbb{E}\left(\sup_{t \in [0, T]} \|Z(t)\|_{L^2}^2\right), \end{aligned}$$

here we have used Assumption (H.4). Hence,

$$\mathbb{E}\left(\sup_{t \in [0, T]} \|\Gamma_3(t)\|_{L^2}^2\right) \leq \varrho T \mathbb{E}\left(\|u_1 - u_2\|_{\Upsilon_T^u}^2\right). \quad (3.21)$$

To estimate Γ_1 , set

$$J_1(s) = \|\theta_m(\|u_1\|_{\Upsilon_s^u})(u_1 \cdot \nabla)u_1 - \theta_m(\|u_2\|_{\Upsilon_s^u})(u_2 \cdot \nabla)u_2\|_{L^2}.$$

We will bound J_1 in four different cases. Set $B(u) = (u \cdot \nabla)u$.

(1) Suppose $\|u_1\|_{\Upsilon_s^u} \vee \|u_2\|_{\Upsilon_s^u} \leq 2m$. From the definition of θ_m , we get

$$\begin{aligned} J_1(s) &\leq \|B(u_1) - B(u_2)\|_{L^2} + \left| \theta_m(\|u_1\|_{\Upsilon_s^u}) - \theta_m(\|u_2\|_{\Upsilon_s^u}) \right| \|B(u_2)\|_{L^2} \\ &\leq \varrho \left(\|u_1\|_{\infty} \|\nabla(u_1 - u_2)\|_{L^2} + \|u_1 - u_2\|_{\infty} \|\nabla u_2\|_{L^2} \right) \\ &\quad + \varrho \frac{C}{m} \|u_1 - u_2\|_{\Upsilon_s^u} \|u_2\|_{\alpha}^2 \\ &\leq \varrho m \|u_1 - u_2\|_{\Upsilon_s^u}. \end{aligned}$$

(2) Suppose $\|u_1\|_{\Upsilon_s^u} \leq 2m$ and $\|u_2\|_{\Upsilon_s^u} > 2m$. We have

$$\begin{aligned} J_1(s) &= \|\theta_m(\|u_1\|_{\Upsilon_s^u})(u_1 \cdot \nabla)u_1\|_{L^2} \\ &= \left| \theta_m(\|u_1\|_{\Upsilon_s^u}) - \theta_m(\|u_2\|_{\Upsilon_s^u}) \right| \|(u_1 \cdot \nabla)u_1\|_{L^2} \\ &\leq \varrho m \|u_1 - u_2\|_{\Upsilon_s^u}. \end{aligned}$$

(3) Suppose $\|u_1\|_{\Upsilon_s^u} > 2m$ and $\|u_2\|_{\Upsilon_s^u} \leq 2m$. Similar to case (2), we have

$$J_1(s) \leq \varrho m \|u_1 - u_2\|_{\Upsilon_s^u}.$$

(4) Suppose $\|u_1\|_{\Upsilon_s^u} \wedge \|u_2\|_{\Upsilon_s^u} > 2m$. Then

$$J_1(s) = 0.$$

Hence, it follows that for all the cases,

$$\Gamma_1(t) \leq \varrho m \|u_1 - u_2\|_{\Upsilon_t} t^{1-\alpha}. \quad (3.22)$$

Combining (3.19) (3.20) (3.21) and (3.22) together we arrive at

$$\begin{aligned} &\mathbb{E} \left(\|\Phi_3(n_1, c_1, u_1) - \Phi_3(n_2, c_2, u_2)\|_{\Upsilon_T}^2 \right) \\ &\leq \varrho T^{2-2\alpha} \mathbb{E} \left(\|n_1 - n_2\|_{\Upsilon_T^n}^2 \right) + \varrho (T + m^2 T^{2-2\alpha}) \mathbb{E} \left(\|u_1 - u_2\|_{\Upsilon_T^u}^2 \right). \end{aligned} \quad (3.23)$$

By virtue of (3.17) (3.18) and (3.23), one can find constants $\rho, C_m > 0$ such that

$$\|\Phi(n_1, c_1, u_1) - \Phi(n_2, c_2, u_2)\|_{S_T}^2 \leq C_m T^\rho \|(n_1, c_1, u_1) - (n_2, c_2, u_2)\|_{S_T}^2. \quad (3.24)$$

Choose $T = T_m$ such that $C_m T_m^\rho = \frac{1}{2}$. Then Φ is a contraction on the space S_{T_m} . Applying the Banach fixed point theorem, we conclude that there exists a unique element $(n_m, c_m, u_m) \in S_{T_m}$ such that (n_m, c_m, u_m) is a solution of (3.1) for $t \in [0, T_m]$.

Let S_T^1 be the space of all $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted, $C^0(\bar{\mathcal{O}}) \times W^{1,q}(\mathcal{O}) \times D(A^\alpha)$ -valued stochastic processes $(n(t), c(t), u(t)), t \geq 0$ such that

$$\|(n, c, u)\|_{S_T^1}^2 := \mathbb{E} \left(\|n\|_{\Upsilon_T^n}^2 \right) + \mathbb{E} \left(\|c\|_{\Upsilon_T^c}^2 \right) + \mathbb{E} \left(\|u\|_{\Upsilon_T^u}^2 \right) < \infty,$$

and

$$(n, c, u) = (n_m, c_m, u_m) \text{ on } [0, T_m], \quad P\text{-a.s..}$$

Then $(S_T^1, \|\cdot\|_{S_T^1})$ is a Banach space.

We introduce a mapping $\Phi^1 = (\Phi_1^1, \Phi_2^1, \Phi_3^1)$ on S_T^1 by defining

$$\begin{aligned} \Phi_1^1(n, c, u)(t + T_m) &:= e^{t\Delta} n_m(T_m) - \int_{T_m}^{T_m+t} e^{(T_m+t-s)\Delta} \left\{ \theta_m(\|n\|_{\Upsilon_s^n}) \theta_m(\|c\|_{\Upsilon_s^c}) \nabla \cdot (n \nabla c) \right. \\ &\quad \left. + \theta_m(\|u\|_{\Upsilon_s^u}) \theta_m(\|n\|_{\Upsilon_s^n}) \nabla \cdot (un) \right\} (s) ds, \end{aligned}$$

$$\begin{aligned} \Phi_2^1(n, c, u)(t + T_m) &:= e^{t\Delta} c_m(T_m) - \int_{T_m}^{T_m+t} e^{(T_m+t-s)\Delta} \left\{ \theta_m(\|n\|_{\Upsilon_s^n}) \theta_m(\|c\|_{\Upsilon_s^c}) n c \right. \\ &\quad \left. + \theta_m(\|u\|_{\Upsilon_s^u}) \theta_m(\|c\|_{\Upsilon_s^c}) u \cdot \nabla c \right\} (s) ds, \end{aligned}$$

and

$$\begin{aligned}\Phi_3^1(n, c, u)(t + T_m) &:= e^{-tA}u_m(t + T_m) - \int_{T_m}^{T_m+t} e^{-(T_m+t-s)A}\theta_m(\|u\|_{\Upsilon_s^u})\mathcal{P}\{(u(s) \cdot \nabla)u(s)\}ds \\ &\quad - \int_{T_m}^{T_m+t} e^{-(T_m+t-s)A}\theta_m(\|n\|_{\Upsilon_s^n})\mathcal{P}\{n(s)\nabla\phi\}ds + \int_{T_m}^{T_m+t} e^{-(T_m+t-s)A}\sigma(u(s))dW_s.\end{aligned}$$

Observe that the constant T_m does not depend on the initial datum. Repeating the above arguments, we can solve (3.1) for $t \in [T_m, 2T_m], [2T_m, 3T_m], \dots$ and we finally obtain a unique solution $(n_m, c_m, u_m) \in S_T$ of (3.1) for any $T > 0$.

Define

$$\tau_m = \inf\{t > 0, \|n_m\|_{\Upsilon_t^n} \vee \|c_m\|_{\Upsilon_t^c} \vee \|u_m\|_{\Upsilon_t^u} \geq m\}. \quad (3.25)$$

The τ_m is a stopping time. When $m \gg \|n_0\|_\infty \vee \|c_0\|_{1,q} \vee \|u_0\|_\alpha$, we have

$$\mathbb{P}(\tau_m > 0) = 1.$$

By the definition of θ_m , it is seen that $(n_m(t), c_m(t), u_m(t))_{t \in [0, \tau_m]}$ is a local variational solution to the system (1.1). \square

3.2 Uniqueness of Local Solutions

Theorem 3.2 *Suppose that (n_1, c_1, u_1, τ^1) and (n_2, c_2, u_2, τ^2) are two local mild/variational solutions of system (1.1). Set $\tau = \tau^1 \wedge \tau^2$. Then we have*

$$(n_1, c_1, u_1) = (n_2, c_2, u_2) \quad \text{on } [0, \tau). \quad (3.26)$$

Proof. Define

$$\tau_R^i = \inf\{t \geq 0, \|n_i\|_{\Upsilon_t^n} + \|c_i\|_{\Upsilon_t^c} + \|u_i\|_{\Upsilon_t^u} \geq R\} \wedge \tau^i, \quad i = 1, 2,$$

and set $\tau_R = \tau_R^1 \wedge \tau_R^2$.

Notice that $\nabla \cdot (u_i(t)n_i(t)) = u_i(t) \cdot \nabla n_i(t)$ because $\nabla \cdot u_i = 0$, $i = 1, 2$. For all $t \in [0, T \wedge \tau_R]$, we have

$$\begin{aligned}& d\|n_1(t) - n_2(t)\|_{L^2}^2 + 2\|\nabla(n_1(t) - n_2(t))\|_{L^2}^2 dt \\ &= -2\left\langle u_1(t) \cdot \nabla n_1(t) - u_2(t) \cdot \nabla n_2(t), n_1(t) - n_2(t) \right\rangle_{L^2} dt \\ &\quad - 2\left\langle \nabla \cdot (n_1(t)\nabla c_1(t)) - \nabla \cdot (n_2(t)\nabla c_2(t)), n_1(t) - n_2(t) \right\rangle_{L^2} dt \\ &\leq \frac{1}{2}\|\nabla(n_1(t) - n_2(t))\|_{L^2}^2 dt + C\|u_1(t)n_1(t) - u_2(t)n_2(t)\|_{L^2}^2 dt \\ &\quad + C\|n_1(t)\nabla c_1(t) - n_2(t)\nabla c_2(t)\|_{L^2}^2 dt \\ &\leq \frac{1}{2}\|\nabla(n_1(t) - n_2(t))\|_{L^2}^2 dt + C\|u_1(t) - u_2(t)\|_{L^2}^2 \|n_1(t)\|_\infty^2 dt \\ &\quad + C\|n_1(t) - n_2(t)\|_{L^2}^2 \|u_2(t)\|_\infty^2 dt + C\|n_1(t)\|_\infty^2 \|\nabla(c_1(t) - c_2(t))\|_{L^2}^2 dt \\ &\quad + C\|\nabla c_2(t)\|_{L^q}^2 \|n_1(t) - n_2(t)\|_{L^{\frac{2q}{q-2}}}^2 dt \\ &\leq \frac{1}{2}\|\nabla(n_1(t) - n_2(t))\|_{L^2}^2 dt + C_R\|n_1(t) - n_2(t)\|_{L^{\frac{2q}{q-2}}}^2 dt \\ &\quad + C_R\left(\|u_1(t) - u_2(t)\|_{L^2}^2 + \|n_1(t) - n_2(t)\|_{L^2}^2 + \|\nabla(c_1(t) - c_2(t))\|_{L^2}^2\right) dt \\ &\leq \|\nabla(n_1(t) - n_2(t))\|_{L^2}^2 dt \\ &\quad + C_R\left(\|u_1(t) - u_2(t)\|_{L^2}^2 + \|n_1(t) - n_2(t)\|_{L^2}^2 + \|\nabla(c_1(t) - c_2(t))\|_{L^2}^2\right) dt. \quad (3.27)\end{aligned}$$

Here for the last inequality, we have used Ehrling's lemma and the compact embedding $W^{1,2}(\mathcal{O}) \hookrightarrow L^{\frac{2q}{q-2}}(\mathcal{O})$.

Recall that $W^{1,q}(\mathcal{O})$ is continuously embedded into $C^0(\bar{\mathcal{O}})$. For all $t \in [0, T \wedge \tau_R)$, we have

$$\begin{aligned}
& d\|c_1(t) - c_2(t)\|_{L^2}^2 + 2\|\nabla(c_1(t) - c_2(t))\|_{L^2}^2 dt \\
&= -2\left\langle u_1(t) \cdot \nabla c_1(t) - u_2(t) \cdot \nabla c_2(t), c_1(t) - c_2(t) \right\rangle_{L^2} dt \\
&\quad -2\left\langle n_1(t)c_1(t) - n_2(t)c_2(t), c_1(t) - c_2(t) \right\rangle_{L^2} dt \\
&\leq \|\nabla(c_1(t) - c_2(t))\|_{L^2}^2 dt + C\|u_1(t)c_1(t) - u_2(t)c_2(t)\|_{L^2}^2 dt \\
&\quad + \|c_1(t) - c_2(t)\|_{L^2}^2 dt + \|n_1(t)c_1(t) - n_2(t)c_2(t)\|_{L^2}^2 dt \\
&\leq \|\nabla(c_1(t) - c_2(t))\|_{L^2}^2 dt + C\left[\|u_1(t)\|_\infty^2 \|c_1(t) - c_2(t)\|_{L^2}^2 + \|c_2(t)\|_\infty^2 \|u_1(t) - u_2(t)\|_{L^2}^2\right] dt \\
&\quad + \|c_1(t) - c_2(t)\|_{L^2}^2 dt + \left[\|n_1(t)\|_\infty^2 \|c_1(t) - c_2(t)\|_{L^2}^2 + \|c_2(t)\|_\infty^2 \|n_1(t) - n_2(t)\|_{L^2}^2\right] dt \\
&\leq \|\nabla(c_1(t) - c_2(t))\|_{L^2}^2 dt + C_R\left[\|c_1(t) - c_2(t)\|_{L^2}^2 + \|n_1(t) - n_2(t)\|_{L^2}^2 + \|u_1(t) - u_2(t)\|_{L^2}^2\right] dt. \tag{3.28}
\end{aligned}$$

By Itô's formula, for all $t \in [0, T \wedge \tau_R)$,

$$\begin{aligned}
& d\|u_1(t) - u_2(t)\|_{L^2}^2 + 2\|\nabla(u_1(t) - u_2(t))\|_{L^2}^2 dt \\
&= -2\left\langle \mathcal{P}\left\{ (u_1(t) \cdot \nabla)u_1(t) - (u_2(t) \cdot \nabla)u_2(t) \right\}, u_1(t) - u_2(t) \right\rangle_{L^2} dt \\
&\quad -2\left\langle \mathcal{P}\{n_1(t)\nabla\phi - n_2(t)\nabla\phi\}, u_1(t) - u_2(t) \right\rangle_{L^2} dt \\
&\quad + 2\left\langle \sigma(u_1(t)) - \sigma(u_2(t)), u_1(t) - u_2(t) \right\rangle_{L^2} dW_t + \|\sigma(u_1(t)) - \sigma(u_2(t))\|_{\mathcal{L}_0^2}^2 dt \\
&\leq \|\nabla(u_1(t) - u_2(t))\|_{L^2}^2 dt \\
&\quad + C\|u_2(t)\|_\infty^4 \|u_1(t) - u_2(t)\|_{L^2}^2 dt \\
&\quad + C\|u_1(t) - u_2(t)\|_{L^2}^2 dt + C\|n_1(t) - n_2(t)\|_{L^2}^2 dt \\
&\quad + 2\left\langle (\sigma(u_1(t)) - \sigma(u_2(t))), u_1(t) - u_2(t) \right\rangle_{L^2} dW_t \\
&\leq \|\nabla(u_1(t) - u_2(t))\|_{L^2}^2 dt + C_R\|u_1(t) - u_2(t)\|_{L^2}^2 dt + C\|n_1(t) - n_2(t)\|_{L^2}^2 dt \\
&\quad + 2\left\langle (\sigma(u_1(t)) - \sigma(u_2(t))), u_1(t) - u_2(t) \right\rangle_{L^2} dW_t. \tag{3.29}
\end{aligned}$$

For the first inequality of (3.29), we have used

$$\begin{aligned}
& \left| \left\langle \mathcal{P}\left\{ (u_1(t) \cdot \nabla)u_1(t) - (u_2(t) \cdot \nabla)u_2(t) \right\}, u_1(t) - u_2(t) \right\rangle_{L^2} \right| \\
&\leq \frac{1}{2}\|\nabla(u_1(t) - u_2(t))\|_{L^2}^2 + C\|u_2(t)\|_{L^4}^4 \|u_1(t) - u_2(t)\|_{L^2}^2 \\
&\leq \frac{1}{2}\|\nabla(u_1(t) - u_2(t))\|_{L^2}^2 + C\|u_2(t)\|_\infty^4 \|u_1(t) - u_2(t)\|_{L^2}^2.
\end{aligned}$$

Combining (3.27) (3.28) and (3.29), we get

$$\begin{aligned}
& \|n_1(t) - n_2(t)\|_{L^2}^2 + C_R\|c_1(t) - c_2(t)\|_{L^2}^2 + \|u_1(t) - u_2(t)\|_{L^2}^2 \\
&\leq \tilde{C}_R \int_0^t \left(\|n_1(s) - n_2(s)\|_{L^2}^2 + \|c_1(s) - c_2(s)\|_{L^2}^2 + \|u_1(s) - u_2(s)\|_{L^2}^2 \right) ds \\
&\quad + 2\left| \int_0^t \left\langle \sigma(u_1(s)) - \sigma(u_2(s)), u_1(s) - u_2(s) \right\rangle_{L^2} dW_s \right|, \quad \text{for all } t \in [0, T \wedge \tau_R). \tag{3.30}
\end{aligned}$$

Let

$$\begin{aligned}
\Theta(T) &:= \mathbb{E}\left(\sup_{t \in [0, T \wedge \tau_R)} \|n_1(t) - n_2(t)\|_{L^2}^2\right) + C_R \mathbb{E}\left(\sup_{t \in [0, T \wedge \tau_R)} \|c_1(t) - c_2(t)\|_{L^2}^2\right) \\
&\quad + \mathbb{E}\left(\sup_{t \in [0, T \wedge \tau_R)} \|u_1(t) - u_2(t)\|_{L^2}^2\right).
\end{aligned}$$

Apply BDG inequality, Assumption (H.3) and Gronwall's lemma to conclude from (3.30) that

$$\Theta(T) = 0.$$

We obtain the uniqueness by noting $\tau_R \uparrow \tau$ as $R \uparrow \infty$. \square

3.3 Global Existence

Definition 3.2 Let (n, c, u, τ) be a local mild/variational solution of system (1.1). If $\limsup_{t \nearrow \tau} (\|n\|_{\Upsilon_t^n} + \|c\|_{\Upsilon_t^c} + \|u\|_{\Upsilon_t^u}) = \infty$ on $\{\omega, \tau < \infty\}$ a.s., then the local solution (n, c, u, τ) is called a maximal local solution.

Recall the stopping times $\{\tau_m, m \in \mathbb{N}\}$ defined in (3.25). By the uniqueness of local solution we proved in Section 3.2, we infer that $\tau_m \leq \tau_{m+1}$ a.s. and

$$(n_{m+1}, c_{m+1}, u_{m+1}) = (n_m, c_m, u_m) \text{ on } [0, \tau_m).$$

Introduce a stopping time:

$$\tau = \lim_{m \rightarrow \infty} \tau_m,$$

and define a stochastic process (n, c, u) on $[0, \tau)$ by

$$(n, c, u) = (n_m, c_m, u_m) \text{ on } t \in [0, \tau_m).$$

Since $\|n_m\|_{\Upsilon_{\tau_m}^n} \vee \|c_m\|_{\Upsilon_{\tau_m}^c} \vee \|u_m\|_{\Upsilon_{\tau_m}^u} \geq m$ on $\{\omega, \tau < \infty\}$, we have

$$\limsup_{t \uparrow \tau} (\|n\|_{\Upsilon_t^n} + \|c\|_{\Upsilon_t^c} + \|u\|_{\Upsilon_t^u}) \geq \lim_{m \uparrow \infty} (\|n_m\|_{\Upsilon_{\tau_m}^n} + \|c_m\|_{\Upsilon_{\tau_m}^c} + \|u_m\|_{\Upsilon_{\tau_m}^u}) = \infty \text{ on } \{\omega, \tau < \infty\}.$$

Therefore (n, c, u, τ) is a maximal local solution of system (1.1).

To obtain the global existence of the solution of the system (1.1), we will establish some a priori estimates for (n, c, u) in the space $L^\infty([0, T \wedge \tau]; C^0(\bar{\mathcal{O}}) \times W^{1,q}(\mathcal{O}) \times D(A^\alpha))$ for any $T > 0$. We first recall the following results from Corollary 4.2 and Lemma 4.5 in [20].

Lemma 3.1 Let $p > 1$ and $r \in [1, \frac{p}{p-1}]$. Then there exists a constant C_T and C_p such that

$$\int_0^{T \wedge \tau} \|n(t, \cdot)\|_{L^p}^r dt \leq C_T \left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t, x)|^2}{n(t, x)} dx dt + 1 \right)^{\frac{(p-1)r}{p}}, \quad (3.31)$$

and

$$\int_{\mathcal{O}} n^p(t, x) dx \leq \left(\int_{\mathcal{O}} n^p(0, x) dx + 1 \right) e^{C_p \int_0^t \int_{\mathcal{O}} |\nabla c(s, x)|^4 dx ds}, \quad t \in [0, T \wedge \tau). \quad (3.32)$$

We start with an estimate of the L^2 norm of u and ∇u .

Lemma 3.2 Let $\theta \in (0, 1)$. Then there exists a constant $C_{T, \|u_0\|_{L^2}}$ such that

$$\begin{aligned} & E \left[\sup_{0 \leq t \leq T \wedge \tau} \|u(t)\|_{L^2}^4 \right] + E \left[\left(\int_0^{T \wedge \tau} \|\nabla u(t)\|_{L^2}^2 dt \right)^2 \right] \\ & \leq C_{T, \|u_0\|_{L^2}} E \left[\left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t, x)|^2}{n(t, x)} dx dt + 1 \right)^\theta \right]. \end{aligned} \quad (3.33)$$

Moreover,

$$E \left[\int_0^{T \wedge \tau} \int_{\mathcal{O}} |u(t, x)|^4 dx dt \right] \leq C_{T, \|u_0\|_{L^2}} E \left[\left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t, x)|^2}{n(t, x)} dx dt + 1 \right)^\theta \right]. \quad (3.34)$$

Proof. By Ito's formula,

$$\begin{aligned} & \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds - \|u_0\|_{L^2}^2 \\ & = -2 \int_0^t \langle u(s), \mathcal{P}\{n(s) \nabla \phi\} \rangle_{L^2} ds + 2 \int_0^t \langle u(s), \sigma(u(s)) dW_s \rangle_{L^2} + \int_0^t \|\sigma(u(s))\|_{\mathcal{L}_0^2}^2 ds. \end{aligned} \quad (3.35)$$

Let $p := \frac{4}{4-\theta}$. As in the proof of Lemma 4.3 in [20], writing $p' := \frac{p}{p-1}$, by the Sobolev imbedding $W^{1,2}(\mathcal{O}) \hookrightarrow L^{p'}(\mathcal{O})$, Hölder's inequality, we have, for $t \in [0, T \wedge \tau]$,

$$\begin{aligned}
& 2 \left| \int_0^t \langle u(s), \mathcal{P}\{n(s)\nabla\phi\} \rangle_{L^2} ds \right| \\
& \leq c \int_0^t \|u(s)\|_{L^{p'}} \|n(s)\|_{L^p} ds \\
& \leq c \int_0^t \|\nabla u(s)\|_{L^2} \|n(s)\|_{L^p} ds \\
& \leq \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + c \int_0^t \|n(s)\|_{L^p}^2 ds \\
& \leq \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + C_T \left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t,x)|^2}{n(t,x)} dx dt + 1 \right)^{\frac{\theta}{2}}, \tag{3.36}
\end{aligned}$$

where Lemma 3.1 was used. Substitute (3.36) into (3.35) to obtain

$$\begin{aligned}
& \sup_{0 \leq t < T \wedge \tau} \|u(t)\|_{L^2}^2 + \int_0^{T \wedge \tau} \|\nabla u(s)\|_{L^2}^2 ds \\
& \leq C_{\|u_0\|_{L^2}, T} \left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t,x)|^2}{n(t,x)} dx dt + 1 \right)^{\frac{\theta}{2}} + C \int_0^{T \wedge \tau} (1 + \|u(t)\|_{L^2}^2) dt \\
& \quad + c \sup_{0 \leq t < T \wedge \tau} \left| \int_0^t \langle u(s), \sigma(u(s)) dW_s \rangle_{L^2} \right|. \tag{3.37}
\end{aligned}$$

Squaring the above inequality and taking expectation, by the BDG inequality and Assumption (H.3), we get

$$\begin{aligned}
& E \left[\sup_{0 \leq t < T \wedge \tau} \|u(t)\|_{L^2}^4 \right] + E \left[\left(\int_0^{T \wedge \tau} \|\nabla u(s)\|_{L^2}^2 ds \right)^2 \right] \\
& \leq C_{\|u_0\|_{L^2}, T} E \left[\left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t,x)|^2}{n(t,x)} dx dt + 1 \right)^{\theta} \right] + C_T E \left[\int_0^{T \wedge \tau} (1 + \|u(t)\|_{L^2}^4) dt \right] \tag{3.38}
\end{aligned}$$

To complete the proof (3.33), we apply the Gronwall's inequality.

By the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}
& \int_0^{T \wedge \tau} \int_{\mathcal{O}} |u(t,x)|^4 dx dt \\
& \leq C \int_0^{T \wedge \tau} \|\nabla u(t, \cdot)\|_{L^2}^2 \|u(t, \cdot)\|_{L^2}^2 dt \\
& \leq C \sup_{0 \leq t < T \wedge \tau} \|u(t)\|_{L^2}^4 + \left(\int_0^{T \wedge \tau} \|\nabla u(t)\|_{L^2}^2 dt \right)^2. \tag{3.39}
\end{aligned}$$

The assertion (3.34) follows from (3.33). \square

Corollary 3.1 *Let $\theta \in (0, 1)$. The following statements hold:*

$$E \left[\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t,x)|^2}{n(t,x)} dx dt + 1 \right] < \infty, \tag{3.40}$$

$$E \left[\int_0^{T \wedge \tau} \int_{\mathcal{O}} |\nabla c(t,x)|^4 dx dt \right] \leq C E \left[\left(\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t,x)|^2}{n(t,x)} dx dt + 1 \right)^{\theta} \right]. \tag{3.41}$$

Proof From the proof of Corollary 4.4 in [20], we know that

$$\int_0^{T \wedge \tau} \int_{\mathcal{O}} \frac{|\nabla n(t,x)|^2}{n(t,x)} dx dt + \frac{1}{4} \int_0^{T \wedge \tau} \int_{\mathcal{O}} g(c(t,x)) |D^2 \rho(c(t,x))|^2 dx dt \leq C_1 + C_2 \int_0^{T \wedge \tau} \int_{\mathcal{O}} |u(t,x)|^4 dx dt, \tag{3.42}$$

and

$$\int_0^{T \wedge \tau} \int_{\mathcal{O}} |\nabla c(t, x)|^4 dx dt \leq C_3 \int_0^{T \wedge \tau} \int_{\mathcal{O}} g(c(t, x)) |D^2 \rho(c(t, x))|^2 dx dt,$$

where $g(c) = \frac{k(c)}{\chi(c)}$, $\rho(c) = \int_0^c \frac{d\sigma}{g(\sigma)}$.

Both (3.40) and (3.41) now follows from Lemma 3.2. \square

To proceed, we recall the following inequality obtained in [20]. For any $p > 1$,

$$\int_{\mathcal{O}} n^p(t, x) dx \leq \left(\int_{\mathcal{O}} n^p(0, x) dx + 1 \right) e^{C_p \int_0^t \int_{\mathcal{O}} |\nabla c(s, x)|^4 dx ds}. \quad (3.43)$$

For $R > 0$, define the stopping time T_R by

$$\begin{aligned} T_R = & \inf \{ t > 0; \int_0^t \int_{\mathcal{O}} \frac{|\nabla n(s, x)|^2}{n(s, x)} dx ds > R, \text{ or } \int_0^t \int_{\mathcal{O}} |\nabla c(s, x)|^4 dx ds > R, \\ & \int_0^t \int_{\mathcal{O}} |\nabla u(s, x)|^2 dx ds > R, \text{ or } \|u(t)\|_{L^2} > R \}. \end{aligned} \quad (3.44)$$

Note that $T_R \rightarrow \infty$ a.s. as $R \rightarrow \infty$. Set $u^R(t, x) := u(t \wedge T_R, x)$, $c^R(t, x) := c(t \wedge T_R, x)$ and $n^R(t, x) := n(t \wedge T_R, x)$. The following result is crucial for establishing the global existence.

Proposition 3.1 *For $R > 0$ and $T > 0$, there exists some constant $C_{R,T} > 0$ such that*

$$E \left[\sup_{0 \leq t \leq T} \|n^R(t, \cdot)\|_{L^\infty} \right] + E \left[\sup_{0 \leq t \leq T} \|\nabla c^R(t, \cdot)\|_{L^p}^2 \right] + E \left[\sup_{0 \leq t \leq T} \|A^\alpha u^R(t, \cdot)\|_{L^2}^2 \right] \leq C_{R,T}. \quad (3.45)$$

Proof. We will prove the proposition along the same lines as in the proof of Theorem 1.1 in [20]. In view of (3.43), for any $p > 1$ we have

$$\int_{\mathcal{O}} |n^R|^p(t, x) dx \leq C_{R,p}. \quad (3.46)$$

Applying Ito's formula and following the similar arguments as in the proofs of (4.16) and (4.17) in [20], we can show that

$$\begin{aligned} & \|\nabla u^R(t)\|_{L^2}^2 + \int_0^{t \wedge T_R} \|\Delta u^R(s)\|_{L^2}^2 ds \\ & \leq C_R + C_R \int_0^{t \wedge T_R} \|\nabla u^R(s)\|_{L^2}^4 ds \\ & \quad + \int_0^{t \wedge T_R} \langle A^{\frac{1}{2}} u^R(s), A^{\frac{1}{2}} \sigma(u^R(s)) dW_s \rangle_{L^2, L^2} + \int_0^{t \wedge T_R} \|\sigma(u^R(s))\|_{\mathcal{L}^2_{\frac{1}{2}}}^2 ds \end{aligned} \quad (3.47)$$

By Gronwall's inequality, it follows that

$$\begin{aligned} & \|\nabla u^R(t)\|_{L^2}^2 + \int_0^{t \wedge T_R} \|\Delta u^R(s)\|_{L^2}^2 ds \\ & \leq \exp \left\{ C_R \int_0^{t \wedge T_R} \|\nabla u^R(s)\|_{L^2}^2 ds \right\} \\ & \quad \times \left[\sup_{0 \leq s \leq t} \left| \int_0^{s \wedge T_R} \langle A^{\frac{1}{2}} u^R(v), A^{\frac{1}{2}} \sigma(u^R(v)) dW_v \rangle_{L^2, L^2} \right| + \int_0^{t \wedge T_R} \|\sigma(u^R(s))\|_{\mathcal{L}^2_{\frac{1}{2}}}^2 ds \right] \end{aligned} \quad (3.48)$$

From the definition of T_R ,

$$\exp \left\{ C_R \int_0^{t \wedge T_R} \|\nabla u^R(s)\|_{L^2}^2 ds \right\} \leq e^{C_R R}.$$

Hence, it follows from the Burkholder inequality and (3.48) that for any $p > 1$,

$$\begin{aligned} & E \left[\sup_{0 \leq s \leq t} \|\nabla u^R(t)\|_{L^2}^{4p} \right] + E \left[\left(\int_0^{t \wedge T_R} \|\Delta u^R(s)\|_{L^2}^2 ds \right)^{2p} \right] \\ & \leq C_R E \left[\int_0^t |\langle A^{\frac{1}{2}} u^R(v), A^{\frac{1}{2}} \sigma(u^R(v)) \rangle_{L^2, L^2}|^2 dv \right]^p + C_R E \left[\int_0^t \|\sigma(u^R(s))\|_{\mathcal{L}^2_{\frac{1}{2}}}^{4p} ds \right] \\ & \leq C_R \left(1 + \int_0^t E [\|\nabla u^R(s)\|_{L^2}^{4p}] ds \right), \end{aligned} \quad (3.49)$$

where we have used the fact that $\|A^{\frac{1}{2}}u\|_{L^2}$ is equivalent to $\|\nabla u^R(s)\|_{L^2}$. By Gronwall's inequality, we get from (3.49) that

$$E[\sup_{0 \leq s \leq T} \|\nabla u^R(t)\|_{L^2}^{4p}] + E[(\int_0^{T \wedge T_R} \|\Delta u^R(s)\|_{L^2}^2 ds)^{2p}] \leq C_{R,p}. \quad (3.50)$$

Next we show that

$$E[\sup_{0 \leq t \leq T} \|A^\alpha u^R(t, \cdot)\|_{L^2}^2] < \infty.$$

By the variation of constants formula we have

$$\begin{aligned} A^\alpha u(t) &= A^\alpha e^{-tA} u_0 + \int_0^t A^\alpha e^{-(t-s)A} n(s) \nabla \phi ds + \int_0^t A^\alpha e^{-(t-s)A} (u(s) \cdot \nabla) u(s) ds \\ &\quad + \int_0^t A^\alpha e^{-(t-s)A} \sigma(u(s)) dW_s \\ &:= A^\alpha e^{-tA} u_0 + u_1(t) + u_2(t) + u_3(t). \end{aligned} \quad (3.51)$$

Clearly,

$$\|A^\alpha e^{-(t \wedge T_R)A} u_0\|_{L^2} \leq \|A^\alpha u_0\|_{L^2}. \quad (3.52)$$

By virtue of (3.46), we have

$$\begin{aligned} \|u_1(t \wedge T_R)\|_{L^2} &\leq \int_0^{t \wedge T_R} \|A^\alpha e^{-(t \wedge T_R - s)A} n^R(s) \nabla \phi\|_{L^2} ds \\ &\leq C \int_0^{t \wedge T_R} (t \wedge T_R - s)^{-\alpha} \|n^R(s)\|_{L^2} ds \leq C(t \wedge T_R)^{1-\alpha}. \end{aligned} \quad (3.53)$$

By Hölder's inequality and Gagliardo-Nirenberg inequality, it holds that

$$\begin{aligned} \|u_2(t \wedge T_R)\|_{L^2} &\leq \int_0^{t \wedge T_R} \|A^\alpha e^{-(t \wedge T_R - s)A} (u^R(s) \cdot \nabla) u^R(s)\|_{L^2} ds \\ &\leq C \int_0^{t \wedge T_R} (t \wedge T_R - s)^{-\alpha} \|(u^R(s) \cdot \nabla) u^R(s)\|_{L^2} ds \\ &\leq C (\int_0^{t \wedge T_R} (t \wedge T_R - s)^{-p' \alpha} ds)^{\frac{1}{p'}} (\int_0^{t \wedge T_R} \|(u^R(s) \cdot \nabla) u^R(s)\|_{L^2}^p ds)^{\frac{1}{p}} \\ &\leq C t^{\frac{1}{p'} - \alpha} (\int_0^{t \wedge T_R} \|\nabla u^R(s)\|_{L^2}^{2p-2} \|\Delta u^R(s)\|_{L^2}^2 ds)^{\frac{1}{p}} \\ &\leq C \sup_{0 \leq s \leq t} \|\nabla u^R(s)\|_{L^2}^{2-\frac{2}{p}} (\int_0^{t \wedge T_R} \|\Delta u^R(s)\|_{L^2}^2 ds)^{\frac{1}{p}}. \end{aligned} \quad (3.54)$$

This along with (3.50) yields

$$E[\sup_{0 \leq t \leq T} \|u_2(t \wedge T_R)\|_{L^2}] \leq C_{T,R}, \quad (3.55)$$

where $C_{T,R}$ is some constant.

To estimate u_3 in (3.51), we notice that u_3 satisfies the SPDE:

$$u_3(t) = - \int_0^t A u_3(s) ds + \int_0^t A^\alpha \sigma(u(s)) dW_s$$

Applying the Ito formula, we get

$$\begin{aligned} &\|u_3(t \wedge T_R)\|_{L^2}^2 + 2 \int_0^{t \wedge T_R} \langle A u_3(s), u_3(s) \rangle_{L^2, L^2} ds \\ &= 2 \int_0^{t \wedge T_R} \langle A^\alpha \sigma(u^R(s)) dW_s, u_3(s \wedge T_R) \rangle_{L^2, L^2} + \int_0^{t \wedge T_R} \|\sigma(u^R(s))\|_{\mathcal{L}_\alpha^2}^2 ds \end{aligned} \quad (3.56)$$

By Burkholder inequality we get from (3.56) that

$$\begin{aligned}
& E[\sup_{0 \leq s \leq t} \|u_3(s \wedge T_R)\|_{L^2}^2] \\
& \leq CE[(\int_0^{t \wedge T_R} \langle A^\alpha \sigma(u^R(s)), u_3(s \wedge T_R) \rangle_{L^2, L^2}^2 ds)^{\frac{1}{2}}] \\
& \quad + E[\int_0^{t \wedge T_R} \|\sigma(u^R(s))\|_{\mathcal{L}_\alpha^2}^2 ds] \\
& \leq \frac{1}{2} E[\sup_{0 \leq s \leq t} \|u_3(s \wedge T_R)\|_{L^2}^2] + CE[\int_0^{t \wedge T_R} \|\sigma(u^R(s))\|_{\mathcal{L}_\alpha^2}^2 ds] \\
& \leq \frac{1}{2} E[\sup_{0 \leq s \leq t} \|u_3(s \wedge T_R)\|_{L^2}^2] + Ct + CE[\int_0^t \|A^\alpha u^R(s)\|_{L^2}^2 ds], \tag{3.57}
\end{aligned}$$

which leads to

$$E[\sup_{0 \leq s \leq t} \|u_3(s \wedge T_R)\|_{L^2}^2] \leq Ct + CE[\int_0^t \|A^\alpha u^R(s)\|_{L^2}^2 ds]. \tag{3.58}$$

Combining (3.51), (3.53), (3.55) and (3.58) together we deduce that

$$E[\sup_{0 \leq s \leq t} \|A^\alpha u(s \wedge T_R)\|_{L^2}^2] \leq C + Ct + CE[\int_0^t \|A^\alpha u^R(s)\|_{L^2}^2 ds]. \tag{3.59}$$

An application of Gronwall's inequality yields

$$E[\sup_{0 \leq t \leq T} \|A^\alpha u^R(t, \cdot)\|_{L^2}^2] < \infty. \tag{3.60}$$

To bound $\|\nabla c^R(t)\|_{L^q}$, we use the variation of constants formula

$$c^R(t) = e^{t \wedge T_R \Delta} c_0 - \int_0^{t \wedge T_R} e^{(t \wedge T_R - s) \Delta} u^R(s) \cdot \nabla c(s) ds - \int_0^{t \wedge T_R} e^{(t \wedge T_R - s) \Delta} k(c(s)) n^R(s) ds \tag{3.61}$$

to obtain

$$\begin{aligned}
& \|\nabla c^R(t)\|_{L^q} \\
& \leq C \|\nabla c_0\|_{L^q} + \int_0^{t \wedge T_R} (t \wedge T_R - s)^{-\frac{1}{2} - (\frac{1}{2} - \frac{1}{q})} \|u^R(s) \cdot \nabla c(s) + k(c(s)) n^R(s)\|_{L^2} ds \\
& \leq C + \int_0^{t \wedge T_R} (t \wedge T_R - s)^{-\frac{1}{2} - (\frac{1}{2} - \frac{1}{q})} (\|u^R(s)\|_{L^\infty} \|\nabla c^R(s)\|_{L^2} + \|n^R(s)\|_{L^2}) ds. \tag{3.62}
\end{aligned}$$

We note that $\|n^R(s)\|_{L^2} \leq C_R$ according to (3.46). On the other hand, by Lemma 3.4 in [20] and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\|\nabla c^R(s)\|_{L^2}^2 & \leq c \int_0^{s \wedge T_R} \|u^R(v)\|_{L^4}^4 dv \leq c \int_0^{s \wedge T_R} \|\nabla u^R(v)\|_{L^2}^2 \cdot \|u^R(v)\|_{L^2}^2 dv \\
& \leq C_R, \tag{3.63}
\end{aligned}$$

where the last inequality follows from the definition of T_R . (3.62) and (3.63) together yields

$$\sup_{0 \leq t \leq T} \|\nabla c^R(t)\|_{L^q} \leq C_{T,R} \tag{3.64}$$

To estimate $\|n^R(t)\|_{L^\infty}$, we fix $\beta \in (\frac{1}{q}, \frac{1}{2})$ and then $r \in (\frac{1}{\beta}, q)$. As the proof of (4.26) in [20], we have

$$\begin{aligned}
\|n^R(t)\|_{L^\infty} & \leq C \int_0^{t \wedge T_R} (t \wedge T_R - s)^{-\frac{1}{2} - \beta} \|n^R \nabla c^R(s) + n^R u^R(s)\|_{L^r} ds \\
& \leq C t^{\frac{1}{2} - \beta} \sup_{0 \leq s \leq t} \{ \|n^R(s)\|_{L^{\frac{qr}{q-r}}} \|\nabla c^R(s)\|_{L^q} + \|n^R\|_{L^r} \|u^R(s)\|_{L^\infty} \} \\
& \leq C_{R,T} + C_{R,T} \sup_{0 \leq s \leq t} \|A^\alpha u^R(s)\|_{L^2}, \tag{3.65}
\end{aligned}$$

where (3.46),(3.64) and the imbedding $D(A^\alpha) \hookrightarrow L^\infty$ have been used. Now, we can conclude from (3.60) that

$$E\left[\sup_{0 \leq t \leq T} \|n^R(t)\|_{L^\infty}\right] \leq C_{R,T} \quad (3.66)$$

for some constant $C_{R,T}$. The proof is complete. \square

Theorem 3.3 *Suppose the conditions in Theorem 2.1 are met. Then, the system (1.1) admits a unique global mild/variational solution.*

Proof. Let (n, c, u, τ) be the maximal local solution of system (1.1) obtained in Section 3.3. From Proposition 3.1 we see that for any $T > 0$, $R > 0$,

$$\tau \geq T \wedge T_R$$

Send R, T go to infinity to get the global existence. Uniqueness was proved in Section 3.2 \square

Remark 3.1 *We notice that the unique global mild/variational solution (n, c, u) obtained in Theorem 3.3 is a weak solution in the sense of Definition 2.2. We only need to verify the statement (1) in Definition 2.2.*

In the proof of Theorem 3.3 (see (3.45)), we have, for any $T > 0$,

$$\sup_{0 \leq t \leq T} \|n(t, \cdot)\|_\infty + \sup_{0 \leq t \leq T} \|\nabla c(t, \cdot)\|_{L^p}^2 + \sup_{0 \leq t \leq T} \|A^\alpha u(t, \cdot)\|_{L^2}^2 < \infty, \quad P\text{-a.s.} \quad (3.67)$$

Theorem 3.3, Lemma 3.2 and (3.40) imply that

$$E\left[\sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^4\right] + E\left[\left(\int_0^T \|\nabla u(t)\|_{L^2}^2 dt\right)^2\right] < \infty, \quad (3.68)$$

$$E\left[\int_0^T \int_{\mathcal{O}} |u(t, x)|^4 dx dt\right] < \infty, \quad (3.69)$$

and

$$E\left[\int_0^T \|\nabla \sqrt{n(t)}\|_{L^2}^2 dt\right] = E\left[\int_0^T \int_{\mathcal{O}} \frac{|\nabla n(t, x)|^2}{n(t, x)} dx dt\right] < \infty. \quad (3.70)$$

Combining (3.67)–(3.70) with Lemma 2.1 and the fact that \mathcal{O} is bounded, it is not difficult to deduce that P -a.s.

$$\begin{aligned} n(1 + |x|) &\in L^\infty([0, T], L^1(\mathcal{O})), \quad \nabla \sqrt{n} \in L^2([0, T], L^2(\mathcal{O})), \\ c &\in L^\infty([0, T], L^\infty(\mathcal{O}) \cap H^1(\mathcal{O})) \cap L^2([0, T], H^2(\mathcal{O})), \\ u &\in C([0, T], H) \cap L^2([0, T], V). \end{aligned} \quad (3.71)$$

For example, to prove $c \in L^\infty([0, T], L^\infty(\mathcal{O}) \cap H^1(\mathcal{O})) \cap L^2([0, T], H^2(\mathcal{O}))$, one can follow the proof of Lemma 4.2 below.

We now estimate $\int_{\mathcal{O}} n |\ln n| dx$. Since

$$\int_{\mathcal{O}} n \ln n dx = \int_{\mathcal{O}} n |\ln n| dx - 2 \int_{\mathcal{O}} n \ln \frac{1}{n} \mathcal{I}_{n \leq 1} dx,$$

and, in view of (2.5),

$$\begin{aligned} 0 \leq \int_{\mathcal{O}} n \ln \frac{1}{n} \mathcal{I}_{n \leq 1} dx &\leq C \int_{\mathcal{O}} n^{1/2} \mathcal{I}_{n \leq 1} dx \\ &\leq C_{\mathcal{O}} \left(\int_{\mathcal{O}} n dx\right)^{1/2} \\ &= C_{\mathcal{O}} \|n_0\|_{L^1}^{1/2}, \end{aligned}$$

it follows that

$$0 \leq \int_{\mathcal{O}} n |\ln n| dx \leq \int_{\mathcal{O}} n \ln n dx + C. \quad (3.72)$$

Recall $\Psi(c) = \int_0^c \sqrt{\frac{\chi(s)}{k(s)}} ds$ in Assumption (B). Lemma 3.4 in [20] implies that

$$\begin{aligned} & \sup_{t \in [0, T]} \left[\int_{\mathcal{O}} n(t, x) \ln n(t, x) dx + \frac{1}{2} \int_{\mathcal{O}} |\nabla \Psi(c(t, x))|^2 dx \right] \\ & \leq \int_{\mathcal{O}} n_0(x) \ln n_0(x) dx + \frac{1}{2} \int_{\mathcal{O}} |\nabla \Psi(c_0(x))|^2 dx + C \int_0^T \int_{\mathcal{O}} |u(t, x)|^4 dx dt. \end{aligned} \quad (3.73)$$

The assumption on n_0, c_0 (see (2.4)) implies that

$$\int_{\mathcal{O}} n_0(x) \ln n_0(x) dx + \frac{1}{2} \int_{\mathcal{O}} |\nabla \Psi(c_0(x))|^2 dx < \infty. \quad (3.74)$$

Putting (3.69) and (3.72)–(3.74) together,

$$n |\ln n| \in L^\infty([0, T], L^1(\mathcal{O})), \quad P\text{-a.s.} \quad (3.75)$$

(3.71) and (3.75) show that the statement (1) in Definition 2.2 holds.

4 Existence and Uniqueness of Weak Solutions

In this part, we assume that conditions (A)–(C) introduced in Section 2 are in place. Our aim is to prove the existence and uniqueness of a weak solution to the system (1.1).

Because the operator A is positive self-adjoint with compact resolvent, there is a complete orthonormal basis $\{e_i, i \in \mathbb{N}\}$ in H made of eigenvectors of A , with corresponding eigenvalues $0 < \beta_i \uparrow \infty$, that is

$$Ae_i = \beta_i e_i, \quad i = 1, 2, \dots$$

4.1 Entropy Function

Let $y \in L^2(\Omega; L^\infty([0, T], H))$ be an adapted process. Let (n, c, u) be a solution to the following system

$$\begin{aligned} dn + u \cdot \nabla n dt &= \delta \Delta n dt - \nabla \cdot (\chi(c) n \nabla c) dt, \\ dc + u \cdot \nabla c dt &= \mu \Delta c dt - k(c) n dt, \\ du + (u \cdot \nabla) u dt + \nabla P dt &= \nu \Delta u dt - n \nabla \phi dt + \sigma(y) dW_t, \\ \nabla \cdot u &= 0, \quad t > 0, \quad x \in \mathcal{O}. \end{aligned} \quad (4.1)$$

Recall $\Psi(c) = \int_0^c \sqrt{\frac{\chi(s)}{k(s)}} ds$ and C_M is the constant appeared in Condition (B). Set

$$\begin{aligned} & \mathcal{E}(n, c, u)(t) \\ &= \int_{\mathcal{O}} n(t) \ln n(t) dx + \frac{1}{2} \|\nabla \Psi(c(t))\|_{L^2}^2 + 2\delta \int_0^t \|\nabla \sqrt{n(s)}\|_{L^2}^2 ds + \frac{K}{\nu} \|u(t)\|_{L^2}^2 + K \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \\ &+ \int_0^t \mu \sum_{i,j} \int_{\mathcal{O}} \left| \partial_i \partial_j \Psi - \frac{d}{dc} \sqrt{\frac{k(c)}{\chi(c)}} \partial_i \Psi \partial_j \Psi \right|^2 dx ds \\ &+ \int_0^t \lambda_1 \mu \int_{\mathcal{O}} |\nabla \Psi|^4 dx ds + 2\lambda_0 \int_0^t \int_{\mathcal{O}} n |\nabla \Psi|^2 dx ds, \end{aligned}$$

here

$$\begin{aligned} 2\lambda_0 &:= \min_{c \in [0, C_M]} \frac{(\chi(c)k(c))'}{2\chi(c)}, \\ 2\lambda_1 &:= \min_{c \in [0, C_M]} -\frac{1}{2} \frac{d^2}{dc^2} \left(\frac{k(c)}{\chi(c)} \right), \end{aligned}$$

are positive by Condition (A).

We have the following result.

Proposition 4.1 *It holds that*

$$d\mathcal{E}(n, c, u)(t) \leq Cdt + C\|u(t)\|_{L^2}^2 dt + C\langle \sigma(y(t)), u(t) \rangle_{L^2} dW_t + C\|\sigma(y(t))\|_{\mathcal{L}_0^2}^2 dt. \quad (4.2)$$

Proof. Lemma 2.1 and Condition (B) imply that c and n preserve the nonnegativity of the initial data, moreover, $\|c(t, \cdot)\|_\infty \leq C_M$.

Keeping in mind the boundary condition (1.2), as (3.5) in [11], we can show that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{O}} n \ln n dx + \frac{1}{2} \frac{d}{dt} \|\nabla \Psi(c)\|_{L^2}^2 + 4\delta \|\nabla \sqrt{n}\|_{L^2}^2 \\ & + \mu \sum_{i,j} \int_{\mathcal{O}} \left| \partial_i \partial_j \Psi - \frac{d}{dc} \sqrt{\frac{k(c)}{\chi(c)}} \partial_i \Psi \partial_j \Psi \right|^2 dx + \lambda_1 \mu \int_{\mathcal{O}} |\nabla \Psi|^4 dx + 2\lambda_0 \int_{\mathcal{O}} n |\nabla \Psi|^2 dx \\ & \leq K \|\nabla u\|_{L^2}^2. \end{aligned} \quad (4.3)$$

Now applying Itô's formula to $\|u\|_{L^2}^2$ and using (2.7), it follows that

$$\begin{aligned} & d\|u\|_{L^2}^2 + 2\nu \|\nabla u\|_{L^2}^2 dt \\ & \leq 2\|\nabla \phi\|_\infty \|n\|_{L^2} \|u\|_{L^2} dt + 2\langle \sigma(y) dW_t, u \rangle_{L^2} + \|\sigma(y)\|_{\mathcal{L}_0^2}^2 dt \\ & \leq \delta \frac{\nu}{K} \|\nabla \sqrt{n}\|_{L^2}^2 dt + C\|\nabla \phi\|_\infty^2 \|n_0\|_{L^1} \|u\|_{L^2}^2 dt + C\|\nabla \phi\|_\infty \|n_0\|_{L^1} \|u\|_{L^2} dt \\ & \quad + 2\langle \sigma(y) dW_t, u \rangle_{L^2} + \|\sigma(y)\|_{\mathcal{L}_0^2}^2 dt. \end{aligned}$$

Adding the above inequality to (4.3), we obtain

$$\begin{aligned} & d \int_{\mathcal{O}} n \ln n dx + \frac{1}{2} d\|\nabla \Psi(c)\|_{L^2}^2 + 2\delta \|\nabla \sqrt{n}\|_{L^2}^2 dt + \frac{K}{\nu} d\|u\|_{L^2}^2 + K \|\nabla u\|_{L^2}^2 dt \\ & + \mu \sum_{i,j} \int_{\mathcal{O}} \left| \partial_i \partial_j \Psi - \frac{d}{dc} \sqrt{\frac{k(c)}{\chi(c)}} \partial_i \Psi \partial_j \Psi \right|^2 dx dt + \lambda_1 \mu \int_{\mathcal{O}} |\nabla \Psi|^4 dx dt + 2\lambda_0 \int_{\mathcal{O}} n |\nabla \Psi|^2 dx dt \\ & \leq Cdt + C\|u\|_{L^2}^2 dt + C\langle \sigma(y), u \rangle_{L^2} dW_t + C\|\sigma(y)\|_{\mathcal{L}_0^2}^2 dt. \end{aligned} \quad (4.4)$$

□

4.2 Energy Estimates for Approximating Solutions

In this section, we consider a sequence of approximating solutions and establish some necessary energy estimates for the proof of the tightness.

Let $H_m = \text{span}\{e_1, \dots, e_m\}$ and define $P_m : H \rightarrow H_m$ as

$$P_m y = \sum_{i=1}^m \langle y, e_i \rangle_{H, H} e_i.$$

Set $\sigma_m = P_m \sigma$. Then

$$\|\sigma_m(u)\|_{\mathcal{L}_{1/2}^2}^2 = \|A^{1/2} \sigma_m(u)\|_{\mathcal{L}_0^2}^2 \leq \beta_m \|\sigma_m(u)\|_{\mathcal{L}_0^2}^2 \leq C\beta_m (1 + \|u\|_H^2) \leq C\beta_m (1 + \frac{1}{\beta_1^2} \|u\|_{1/2}^2).$$

Similarly, we can prove that

$$\begin{aligned} & \|\sigma_m(u_1) - \sigma_m(u_2)\|_{\mathcal{L}_{1/2}^2}^2 \leq C_m \|u_1 - u_2\|_{1/2}^2, \\ & \|\sigma_m(u)\|_{\mathcal{L}_\alpha^2}^2 \leq C_m (1 + \|u\|_\alpha^2), \\ & \|\sigma_m(u_1) - \sigma_m(u_2)\|_{\mathcal{L}_\alpha^2}^2 \leq C_m \|u_1 - u_2\|_\alpha^2. \end{aligned}$$

This shows that σ_m satisfies Conditions **(H.3)** **(H.4)** and **(H.5)**.

For any (n_0, c_0, u_0) satisfying Condition (B) in Section 2, it is easy to see that one can find (n_0^m, c_0^m, u_0^m) such that

- (1) (n_0^m, c_0^m, u_0^m) satisfies (2.4),
(2)

$$\begin{aligned} n_0^m(1 + |x| + |\ln n_0^m|) &\rightarrow n_0(1 + |x| + |\ln n_0|) \text{ in } L^1(\mathcal{O}), \\ c_0^m &\leq C_M \text{ and } c_0^m \rightarrow c_0 \text{ in } W^{1,2}(\mathcal{O}), \\ \nabla \Psi(c_0^m) &\rightarrow \nabla \Psi(c_0) \text{ in } L^2(\mathcal{O}), \\ u_0^m &\rightarrow u_0 \text{ in } L^2(\mathcal{O}). \end{aligned}$$

By Theorem 3.3 and Remark 3.1, we know that there exists an adapted $C^0(\bar{\mathcal{O}}) \times W^{1,q}(\mathcal{O}) \times D(A^\alpha)$ -valued stochastic process (n^m, c^m, u^m) satisfying the following SPDE:

$$\begin{aligned} dn^m + u^m \cdot \nabla n^m dt &= \delta \Delta n^m dt - \nabla \cdot (\chi(c^m) n^m \nabla c^m) dt, \\ dc^m + u^m \cdot \nabla c^m dt &= \mu \Delta c^m dt - k(c^m) n^m dt, \\ du^m + (u^m \cdot \nabla) u^m dt + \nabla P dt &= \nu \Delta u^m dt - n^m \nabla \phi dt + \sigma_m(u^m) dW_t, \\ \nabla \cdot u^m &= 0, \quad t > 0, \quad x \in \mathcal{O}, \end{aligned} \tag{4.5}$$

with initial value (n_0^m, c_0^m, u_0^m) .

In the rest of this section, we will provide a number of estimates for (n^m, c^m, u^m) .

Lemma 4.1 *There exists a constant C_T independent of m such that*

$$\mathbb{E} \left(\sup_{t \in [0, T]} \mathcal{E}(n^m, c^m, u^m)(t) \right) \leq C_T \left(\int_{\mathcal{O}} n_0 \ln n_0 dx + \|\nabla \Psi(c_0)\|_{L^2}^2 + \|u_0\|_{L^2}^2 + 1 \right). \tag{4.6}$$

Proof. By (4.4), we have the following estimates

$$\begin{aligned} &\mathcal{E}(n^m, c^m, u^m)(t) \\ &\leq \int_{\mathcal{O}} n_0^m \ln n_0^m(t) dx + \frac{1}{2} \|\nabla \Psi(c_0^m)\|_{L^2}^2 + \frac{K}{\nu} \|u_0^m\|_{L^2}^2 + Ct + C \int_0^t \|u^m(s)\|_{L^2}^2 ds \\ &\quad + C \int_0^t \langle \sigma_m(u^m(s)) dW_s, u^m(s) \rangle_{L^2} + C \int_0^t \|\sigma_m(u^m(s))\|_{\mathcal{L}_0^2}^2 ds. \end{aligned} \tag{4.7}$$

In particular, together with (3.72), we have

$$\frac{K}{\nu} \|u^m(t)\|_{L^2}^2 + K \int_0^t \|\nabla u^m(s)\|_{L^2}^2 ds \tag{4.8}$$

$$\begin{aligned} &\leq \int_{\mathcal{O}} n_0^m \ln n_0^m(t) dx + \frac{1}{2} \|\nabla \Psi(c_0^m)\|_{L^2}^2 + \frac{K}{\nu} \|u_0^m\|_{L^2}^2 + Ct + C \int_0^t \|u^m(s)\|_{L^2}^2 ds \\ &\quad + C \int_0^t \langle \sigma_m(u^m(s)) dW_s, u^m(s) \rangle_{L^2} + C \int_0^t \|\sigma_m(u^m(s))\|_{\mathcal{L}_0^2}^2 ds. \end{aligned} \tag{4.9}$$

By the BDG inequality and the growth condition (C) on σ ,

$$\begin{aligned} &\mathbb{E} \left(\sup_{t \in [0, T]} C \left| \int_0^t \langle \sigma_m(u^m(s)) dW_s, u^m(s) \rangle_{L^2} \right| \right) \\ &\leq C \mathbb{E} \left(\int_0^T \|u^m(s)\|_{L^2}^2 \|\sigma_m(u^m(s))\|_{\mathcal{L}_0^2}^2 ds \right)^{1/2} \\ &\leq 1/2 \frac{K}{\nu} \mathbb{E} \left(\sup_{t \in [0, T]} \|u^m(t)\|_{L^2}^2 \right) + C \mathbb{E} \int_0^T (1 + \|u^m(s)\|_{L^2}^2) ds, \end{aligned} \tag{4.10}$$

and

$$\mathbb{E} \left(C \int_0^T \|\sigma_m(u^m(s))\|_{\mathcal{L}_0^2}^2 ds \right) \leq C \mathbb{E} \int_0^T (1 + \|u^m(s)\|_{L^2}^2) ds. \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.8), and applying the Gronwall's Lemma, we obtain

$$\begin{aligned}
& \mathbb{E}\left(\sup_{t \in [0, T]} \|u^m(t)\|_{L^2}^2\right) + \mathbb{E}\left(\int_0^T \|\nabla u^m(s)\|_{L^2}^2 ds\right) \\
& \leq C\left(\int_{\mathcal{O}} n_0^m \ln n_0^m dx + \|\nabla \Psi(c_0^m)\|_{L^2}^2 + \|u_0^m\|_{L^2}^2 + T\right)e^{CT} \\
& \leq C\left(\int_{\mathcal{O}} n_0 \ln n_0 dx + \|\nabla \Psi(c_0)\|_{L^2}^2 + \|u_0\|_{L^2}^2 + T\right)e^{CT}
\end{aligned} \tag{4.12}$$

(4.7) and (4.12) together imply that

$$\mathbb{E}\left(\sup_{t \in [0, T]} \mathcal{E}(n^m, c^m, u^m)(t)\right) \leq C_T\left(\int_{\mathcal{O}} n_0 \ln n_0 dx + \|\nabla \Psi(c_0)\|_{L^2}^2 + \|u_0\|_{L^2}^2 + 1\right). \tag{4.13}$$

□

Corollary 4.1 *There exists a constant C independent of m such that*

- (a) $0 \leq n^m(t, x)$ and $c^m(t, x) \in [0, C_M]$, for all $t \geq 0$, $x \in \mathcal{O}$,
(b)

$$\mathbb{E}\left(\sup_{t \in [0, T]} n^m(t) \ln n^m(t)\right) \leq C, \tag{4.14}$$

and

$$\mathbb{E}\left(\int_0^T \|\nabla \sqrt{n^m(t)}\|_{L^2}^2 dt\right) \leq C. \tag{4.15}$$

(c)

$$\mathbb{E}\left(\sup_{t \in [0, T]} n^m(t) |\ln n^m(t)|\right) \leq C. \tag{4.16}$$

(d)

$$\mathbb{E}\left(\int_0^T \|n^m(t)\|_{L^2}^2 dt\right) \leq C\mathbb{E}\left(\int_0^T \|n_0^m\|_{L^1} \|\nabla \sqrt{n^m}\|_{L^2}^2 + \|n_0^m\|_{L^1}^2 dt\right) \leq C. \tag{4.17}$$

(e)

$$\mathbb{E}\left(\sup_{t \in [0, T]} \|u^m(t)\|_{L^2}^2\right) + \mathbb{E}\left(\int_0^T \|\nabla u^m(s)\|_{L^2}^2 ds\right) \leq C.$$

Proof. (a) follows from the comparison theorem, see Lemma 2.1. (b) is a consequence of (4.13). (3.72) and (4.14) imply (c). By (2.7) and (4.15), we have

$$\mathbb{E}\left(\int_0^T \|n^m(t)\|_{L^2}^2 dt\right) \leq C\mathbb{E}\left(\int_0^T \|n_0^m\|_{L^1} \|\nabla \sqrt{n^m}\|_{L^2}^2 + \|n_0^m\|_{L^1}^2 dt\right) \leq C. \tag{4.18}$$

(e) is the statement of (4.12). □

For $N \geq 1$, put

$$\Omega_N^m := \left\{ \omega : \sup_{s \in [0, T]} \|u^m(s)\|_{L^2}^2 \bigvee \int_0^T \|\nabla u^m(s)\|_{L^2}^2 ds \bigvee \int_0^T \|n^m(s)\|_{L^2}^2 ds \leq N \right\}.$$

By Corollary 4.1 and the Chebyshev's inequality, we find that

$$\begin{aligned}
& \mathbb{P}(\Omega_N^m) \\
& \geq 1 - \mathbb{P}\left(\sup_{s \in [0, T]} \|u^m(s)\|_{L^2}^2 > N\right) - \mathbb{P}\left(\int_0^T \|\nabla u^m(s)\|_{L^2}^2 ds > N\right) - \mathbb{P}\left(\int_0^T \|n^m(s)\|_{L^2}^2 ds > N\right) \\
& \geq 1 - \frac{3C}{N}.
\end{aligned} \tag{4.19}$$

Lemma 4.2 *We have*

$$\sup_{s \in [0, T]} \|c^m(s)\|_{H^1}^2 + \int_0^T \|c^m(s)\|_{H^2}^2 ds \leq C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}, \quad \omega \in \Omega_N^m, \quad (4.20)$$

where the constant $C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}$ is independent of m .

Proof.

By the chain rule, we have

$$\begin{aligned} & \|c^m(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla c^m(s)\|_{L^2}^2 ds \\ = & \|c^m(0)\|_{L^2}^2 - 2 \int_0^t \langle u^m(s) \cdot \nabla c^m(s), c^m(s) \rangle_{L^2} ds - 2 \int_0^t \langle k(c^m(s))n^m(s), c^m(s) \rangle_{L^2} ds \\ \leq & \|c(0)\|_{L^2}^2 + 2 \int_0^t \langle u^m(s)c^m(s), \nabla c^m(s) \rangle_{L^2} ds + \sup_{r \in [0, C_M]} k^2(r) \int_0^t \|n^m(s)\|_{L^2}^2 ds + \int_0^t \|c^m(s)\|_{L^2}^2 ds \\ \leq & \|c(0)\|_{L^2}^2 + \mu \int_0^t \|\nabla c^m(s)\|_{L^2}^2 ds + \frac{C_M^2}{\mu} \int_0^t \|u^m(s)\|_{L^2}^2 ds \\ & + \sup_{r \in [0, C_M]} k^2(r) \int_0^t \|n^m(s)\|_{L^2}^2 ds + \int_0^t \|c^m(s)\|_{L^2}^2 ds, \end{aligned}$$

where we have used (a) of Corollary 4.1 and $u^m(s) \cdot \nabla c^m(s) = \nabla \cdot (u^m(s)c^m(s))$ (due to $\nabla \cdot u^m(s) = 0$). Hence

$$\begin{aligned} & \|c^m(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla c^m(s)\|_{L^2}^2 ds \\ \leq & \|c(0)\|_{L^2}^2 + \frac{C_M^2}{\mu} \int_0^T \|u^m(s)\|_{L^2}^2 ds + \sup_{r \in [0, C_M]} k^2(r) \int_0^T \|n^m(s)\|_{L^2}^2 ds + \int_0^t \|c^m(s)\|_{L^2}^2 ds. \end{aligned}$$

By the Gronwall's lemma,

$$\begin{aligned} & \sup_{t \in [0, T]} \|c^m(t)\|_{L^2}^2 + \mu \int_0^T \|\nabla c^m(s)\|_{L^2}^2 ds \\ \leq & e^T \cdot \left(\|c(0)\|_{L^2}^2 + \frac{C_M^2}{\mu} \int_0^T \|u^m(s)\|_{L^2}^2 ds + \sup_{r \in [0, C_M]} k^2(r) \int_0^T \|n^m(s)\|_{L^2}^2 ds \right). \end{aligned} \quad (4.21)$$

Hence we have for $\omega \in \Omega_N^m$,

$$\sup_{t \in [0, T]} \|c^m(t)\|_{L^2}^2 + \mu \int_0^T \|\nabla c^m(s)\|_{L^2}^2 ds \leq e^T \cdot \left(\|c(0)\|_{L^2}^2 + \frac{C_M^2}{\mu} TN + \sup_{r \in [0, C_M]} k^2(r)N \right). \quad (4.22)$$

Next we estimate $\|\nabla c^m(t)\|_{L^2}^2$. Again by the chain rule,

$$\begin{aligned} & \|\nabla c^m(t)\|_{L^2}^2 + 2 \int_0^t \langle u^m(s) \cdot \nabla c^m(s), -\Delta c^m(s) \rangle_{L^2} ds \\ = & \|\nabla c^m(0)\|_{L^2}^2 - 2\mu \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds - 2 \int_0^t \langle k(c^m(s))n^m(s), -\Delta c^m(s) \rangle_{L^2} ds \\ \leq & \|\nabla c(0)\|_{L^2}^2 - 2\mu \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds + \frac{2}{\mu} \sup_{r \in [0, C_M]} k^2(r) \int_0^t \|n^m(s)\|_{L^2}^2 ds + \frac{\mu}{2} \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds. \end{aligned}$$

Noticing

$$\left| 2 \int_0^t \langle u^m(s) \cdot \nabla c^m(s), -\Delta c^m(s) \rangle_{L^2} ds \right| \leq \frac{\mu}{2} \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds + \frac{2}{\mu} \int_0^t \|u^m(s) \cdot \nabla c^m(s)\|_{L^2}^2 ds,$$

it follows that

$$\begin{aligned} & \|\nabla c^m(t)\|_{L^2}^2 + \mu \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds \\ & \leq \|\nabla c(0)\|_{L^2}^2 + \frac{2}{\mu} \int_0^t \|u^m(s) \cdot \nabla c^m(s)\|_{L^2}^2 ds + \frac{2}{\mu} \sup_{r \in [0, C_M]} k^2(r) \int_0^t \|n^m(s)\|_{L^2}^2 ds. \end{aligned} \quad (4.23)$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u^m(s)\|_{L^4}^4 \leq 2\|u^m(s)\|_{L^2}^2 \|\nabla u^m(s)\|_{L^2}^2. \quad (4.24)$$

Recall also the Gagliardo-Nirenberg-Sobolev inequality:

$$\|f\|_{L^4} \leq C(\|\nabla f\|_{L^2}^{1/2} \|f\|_{L^2}^{1/2} + \|f\|_{L^2}). \quad (4.25)$$

Hence we can find a constant $C > 0$ such that

$$\|\nabla c^m(s)\|_{L^4} \leq C\left(\|c^m(s)\|_{H^2}^{1/2} \|\nabla c^m(s)\|_{L^2}^{1/2} + \|\nabla c^m(s)\|_{L^2}\right), \quad (4.26)$$

here

$$\|f\|_{H^2}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 + \sum_{i+j=2, i,j \geq 0} \int_{\mathcal{O}} \left| \frac{\partial^i \partial^j f(x_1, x_2)}{\partial x_1^i \partial x_2^j} \right|^2 dx_1 dx_2.$$

According to Proposition 7.2 in [15](Page 404), for any $f \in H^2$ satisfying the Neumann boundary condition one has

$$\|f\|_{H^2}^2 \leq C(\|\Delta f\|_{L^2}^2 + \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2). \quad (4.27)$$

By (4.24) (4.26) and (4.27), for any $\omega \in \Omega_N^m$, we have

$$\begin{aligned} & \frac{2}{\mu} \int_0^t \|u^m(s) \cdot \nabla c^m(s)\|_{L^2}^2 ds \\ & \leq C_\mu \int_0^t \|u^m(s)\|_{L^4}^2 \|\nabla c^m(s)\|_{L^4}^2 ds \\ & \leq C_\mu \int_0^t \|u^m(s)\|_{L^2} \|\nabla u^m(s)\|_{L^2} \left(\|c^m(s)\|_{H^2} \|\nabla c^m(s)\|_{L^2} + \|\nabla c^m(s)\|_{L^2}^2 \right) ds \\ & \leq C_\mu N^{1/2} \int_0^t \|\nabla u^m(s)\|_{L^2} \left(\|\Delta c^m(s)\|_{L^2} + \|\nabla c^m(s)\|_{L^2} + \|c^m(s)\|_{L^2} \right) \|\nabla c^m(s)\|_{L^2} ds \\ & \quad + C_\mu N^{1/2} \int_0^t \|\nabla u^m(s)\|_{L^2} \|\nabla c^m(s)\|_{L^2}^2 ds \\ & \leq \frac{\mu}{2} \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds + C_\mu N \int_0^t \|\nabla u^m(s)\|_{L^2}^2 \|\nabla c^m(s)\|_{L^2}^2 ds \\ & \quad + C_\mu N^{1/2} \int_0^t \|\nabla u^m(s)\|_{L^2} \|c^m(s)\|_{L^2} \|\nabla c^m(s)\|_{L^2} ds + C_\mu N^{1/2} \int_0^t \|\nabla u^m(s)\|_{L^2} \|\nabla c^m(s)\|_{L^2}^2 ds \\ & \leq \frac{\mu}{2} \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds + C_\mu \int_0^t \left(3N \|\nabla u^m(s)\|_{L^2}^2 + 1 \right) \|\nabla c^m(s)\|_{L^2}^2 ds \\ & \quad + C_\mu t \sup_{s \in [0, t]} \|c^m(s)\|_{L^2}^2. \end{aligned} \quad (4.28)$$

For $\omega \in \Omega_N^m$, substituting (4.28) into (4.23) we obtain

$$\begin{aligned} & \|\nabla c^m(t)\|_{L^2}^2 + \frac{\mu}{2} \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds \\ & \leq \|\nabla c(0)\|_{L^2}^2 + \frac{2}{\mu} \sup_{r \in [0, C_M]} k^2(r) N + C_\mu \int_0^t \left(3N \|\nabla u^m(s)\|_{L^2}^2 + 1 \right) \|\nabla c^m(s)\|_{L^2}^2 ds \\ & \quad + C_\mu T \sup_{s \in [0, T]} \|c^m(s)\|_{L^2}^2. \end{aligned}$$

Hence by Gronwall's lemma and (4.22), it follows that

$$\begin{aligned}
& \|\nabla c^m(t)\|_{L^2}^2 + \frac{\mu}{2} \int_0^t \|\Delta c^m(s)\|_{L^2}^2 ds \\
& \leq e^{C_\mu \int_0^T (3N \|\nabla u^m(s)\|_{L^2}^2 + 1) ds} \cdot \left[\|\nabla c(0)\|_{L^2}^2 + \frac{2}{\mu} \sup_{r \in [0, C_M]} k^2(r) N \right. \\
& \quad \left. + C_\mu T e^T \cdot \left(\|c(0)\|_{L^2}^2 + \frac{C_M^2}{\mu} T N + \sup_{r \in [0, C_M]} k^2(r) N \right) \right] \\
& \leq e^{3C_\mu N^2 + C_\mu T} \left[\|\nabla c(0)\|_{L^2}^2 + \frac{2}{\mu} \sup_{r \in [0, C_M]} k^2(r) N \right. \\
& \quad \left. + C_\mu T e^T \cdot \left(\|c(0)\|_{L^2}^2 + \frac{C_M^2}{\mu} T N + \sup_{r \in [0, C_M]} k^2(r) N \right) \right]. \tag{4.29}
\end{aligned}$$

Thus, in view of (4.27) (4.22) and (4.29), we can conclude that

$$\sup_{s \in [0, T]} \|c^m(s)\|_{H^1}^2 + \int_0^T \|c^m(s)\|_{H^2}^2 ds \leq C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}, \quad \omega \in \Omega_N^m. \tag{4.30}$$

The constant $C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}$ is independent of m . \square

Corollary 4.2 *There exists a constant $C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}$ such that for all $\omega \in \Omega_N^m$,*

$$\int_0^T \left\| \frac{dc^m(t)}{dt} \right\|_{L^2}^2 dt \leq C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}. \tag{4.31}$$

Proof. Combining (4.28) and (4.30), for $\omega \in \Omega_N^m$, we have

$$\int_0^T \|u^m(s) \cdot \nabla c^m(s)\|_{L^2}^2 ds \leq C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}.$$

Hence, for $\omega \in \Omega_N^m$, by (4.30), we have

$$\begin{aligned}
& \int_0^T \left\| \frac{dc^m(t)}{dt} \right\|_{L^2}^2 dt \\
& \leq C \left[\int_0^T \|u^m(s) \cdot \nabla c^m(s)\|_{L^2}^2 ds + \mu^2 \int_0^T \|\Delta c^m(s)\|_{L^2}^2 ds + \sup_{r \in [0, C_M]} k^2(r) \int_0^T \|n^m(s)\|_{L^2}^2 ds \right] \\
& \leq C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}, \quad \omega \in \Omega_N^m. \tag{4.32}
\end{aligned}$$

\square

Lemma 4.3 *There exists a constant $C(T, N)$ such that for all $\omega \in \Omega_N^m$ and $m \geq 1$,*

$$\int_0^T \left\| \frac{dn^m(t)}{dt} \right\|_{H^{-3}}^2 dt \leq C(T, N), \tag{4.33}$$

and

$$\sup_{t \in [0, T]} \|n^m(t)\|_{L^2}^2 + \int_0^T \|\nabla n^m(s)\|_{L^2}^2 ds \leq C(T, N). \tag{4.34}$$

Proof.

We first prove (4.33). According to the Sobolev inequalities, we have

$$\|\nabla \varphi\|_\infty \leq C \|\varphi\|_{H^3}, \quad \varphi \in H^3(\mathcal{O}).$$

Hence, for $\varphi \in H^3(\mathcal{O})$,

$$\begin{aligned}
& \left| \left\langle \varphi, \frac{dn^m(t)}{dt} \right\rangle_{L^2} \right| \\
& \leq \delta \left| \langle \Delta \varphi, n^m(t) \rangle_{L^2} \right| + \left| \langle \nabla \varphi, n^m(t) u^m(t) \rangle_{L^2} \right| + \left| \langle \nabla \varphi, \chi(c^m(t)) n^m(t) \nabla c^m(t) \rangle_{L^2} \right| \\
& \leq \delta \|\Delta \varphi\|_{L^2} \|n^m(t)\|_{L^2} + C \|\varphi\|_{H^3} \|n^m(t)\|_{L^2} \|u^m(t)\|_{L^2} + C \|\varphi\|_{H^3} \|n^m(t)\|_{L^2} \|\nabla c^m(t)\|_{L^2}.
\end{aligned}$$

Therefore, for $\omega \in \Omega_N^m$,

$$\begin{aligned}
& \int_0^T \left\| \frac{dn^m(t)}{dt} \right\|_{H^{-3}}^2 dt \\
& \leq C \left[\delta \int_0^T \|n^m(t)\|_{L^2}^2 dt + \int_0^T \|n^m(t)\|_{L^2}^2 \|u^m(t)\|_{L^2}^2 dt + \int_0^T \|n^m(t)\|_{L^2}^2 \|\nabla c^m(t)\|_{L^2}^2 dt \right] \\
& \leq C \left[\delta N + N^2 + N \sup_{t \in [0, T]} \|\nabla c^m(t)\|_{L^2}^2 \right] \\
& \leq C \left[\delta N + N^2 + C_{\mu, T, N, C_M, \|c(0)\|_{H^1}} \right].
\end{aligned} \tag{4.35}$$

(4.30) has been used in the last inequality. This proves (4.33).

By the chain rule,

$$\begin{aligned}
& \|n^m(t)\|_{L^2}^2 + 2\delta \int_0^t \|\nabla n^m(s)\|_{L^2}^2 ds \\
& = \|n^m(0)\|_{L^2}^2 + 2 \int_0^t \langle u^m(s)n^m(s), \nabla n^m(s) \rangle_{L^2} ds + 2 \int_0^t \langle \chi(c^m(s))n^m(s) \nabla c^m(s), \nabla n^m(s) \rangle_{L^2} ds \\
& \leq \|n_0\|_{L^2}^2 + \delta \int_0^t \|\nabla n^m(s)\|_{L^2}^2 ds + \frac{2}{\delta} \int_0^t \|u^m(s)n^m(s)\|_{L^2}^2 ds \\
& \quad + \frac{2}{\delta} \sup_{r \in [0, C_M]} \chi^2(r) \int_0^t \|n^m(s) \nabla c^m(s)\|_{L^2}^2 ds.
\end{aligned} \tag{4.36}$$

By (4.24) and (4.25),

$$\begin{aligned}
& \frac{2}{\delta} \int_0^t \|u^m(s)n^m(s)\|_{L^2}^2 ds \\
& \leq C_\delta \int_0^t \|u^m(s)\|_{L^4}^2 \|n^m(s)\|_{L^4}^2 ds \\
& \leq C_\delta \int_0^t \|u^m(s)\|_{L^2} \|\nabla u^m(s)\|_{L^2} \left(\|\nabla n^m(s)\|_{L^2} \|n^m(s)\|_{L^2} + \|n^m(s)\|_{L^2}^2 \right) ds \\
& \leq \frac{\delta}{4} \int_0^t \|\nabla n^m(s)\|_{L^2}^2 ds + C_\delta \int_0^t \|n^m(s)\|_{L^2}^2 \left(\|u^m(s)\|_{L^2}^2 \|\nabla u^m(s)\|_{L^2}^2 + \|u^m(s)\|_{L^2} \|\nabla u^m(s)\|_{L^2} \right) ds.
\end{aligned} \tag{4.37}$$

In view of (4.25) and (4.26), we have

$$\begin{aligned}
& \frac{2}{\delta} \sup_{r \in [0, C_M]} \chi^2(r) \int_0^t \|n^m(s) \nabla c^m(s)\|_{L^2}^2 ds \\
& \leq C_\delta \int_0^t \|n^m(s)\|_{L^4}^2 \|\nabla c^m(s)\|_{L^4}^2 ds \\
& \leq C_\delta \int_0^t \left(\|\nabla n^m(s)\|_{L^2} \|n^m(s)\|_{L^2} + \|n^m(s)\|_{L^2}^2 \right) \cdot \left(\|c^m(s)\|_{H^2} \|\nabla c^m(s)\|_{L^2} + \|\nabla c^m(s)\|_{L^2}^2 \right) ds \\
& \leq \frac{\delta}{4} \int_0^t \|\nabla n^m(s)\|_{L^2}^2 ds + C_\delta \int_0^t \|n^m(s)\|_{L^2}^2 \left(\|c^m(s)\|_{H^2}^2 \|\nabla c^m(s)\|_{L^2}^2 + \|\nabla c^m(s)\|_{L^2}^4 \right. \\
& \quad \left. + \|c^m(s)\|_{H^2} \|\nabla c^m(s)\|_{L^2} + \|\nabla c^m(s)\|_{L^2}^2 \right) ds.
\end{aligned} \tag{4.38}$$

Combining (4.36)–(4.38), and applying the Gronwall's lemma, we obtain

$$\sup_{t \in [0, T]} \|n^m(t)\|_{L^2}^2 + \frac{\delta}{4} \int_0^T \|\nabla n^m(s)\|_{L^2}^2 ds \leq \|n_0\|_{L^2}^2 e^{C_\delta \Xi(T)}$$

where

$$\begin{aligned}
\Xi(T) : &= \int_0^T \left(\|u^m(s)\|_{L^2}^2 \|\nabla u^m(s)\|_{L^2}^2 + \|u^m(s)\|_{L^2} \|\nabla u^m(s)\|_{L^2} \right. \\
& \quad \left. + \|c^m(s)\|_{H^2}^2 \|\nabla c^m(s)\|_{L^2}^2 + \|\nabla c^m(s)\|_{L^2}^4 \right. \\
& \quad \left. + \|c^m(s)\|_{H^2} \|\nabla c^m(s)\|_{L^2} + \|\nabla c^m(s)\|_{L^2}^2 \right) ds.
\end{aligned}$$

From the definition of Ω_N^m and (4.30), we deduce that

$$\sup_{t \in [0, T]} \|n^m(t)\|_{L^2}^2 + \int_0^T \|\nabla n^m(s)\|_{L^2}^2 ds \leq C_{\mu, \delta, T, N, C_M, \|c(0)\|_{H^1}}, \quad \omega \in \Omega_N^m. \quad (4.39)$$

□

Remark 4.1 (4.19) and (4.39) imply that

$$\sup_{t \in [0, T]} \|n^m(t)\|_{L^2}^2 + \int_0^T \|\nabla n^m(s)\|_{L^2}^2 ds < \infty, \quad P\text{-a.s.}$$

4.3 Existence of Martingale Weak Solutions

Definition 4.1 We say that there exists a martingale weak solution to the system (1.1) if there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ and, on this basis, a U -cylindrical Wiener process W , a progressively measurable process (n, c, u) satisfying

(1) P -a.s.

$$\begin{aligned} n(1 + |x| + |\ln n|) &\in L^\infty([0, T], L^1(\mathcal{O})), \quad \nabla \sqrt{n} \in L^2([0, T], L^2(\mathcal{O})), \\ c &\in L^\infty([0, T], L^\infty(\mathcal{O}) \cap H^1(\mathcal{O})) \cap L^2([0, T], H^2(\mathcal{O})), \\ u &\in C([0, T], H) \cap L^2([0, T], V); \end{aligned}$$

(2) For all $\psi_1, \psi_2 \in C^\infty([0, T] \times \mathcal{O})$ with compact supports with respect to the space variable, and $\psi_1(T, \cdot) = \psi_2(T, \cdot) = 0$, P -a.s.

$$\int_{\mathcal{O}} \psi_1(0, x) n_0 dx = \int_0^T \int_{\mathcal{O}} n [\partial_t \psi_1 + \nabla \psi_1 \cdot u + \delta \Delta \psi_1 + \nabla \psi_1 \cdot (\chi(c) \nabla c)] dx dt,$$

$$\int_{\mathcal{O}} \psi_2(0, x) c_0 dx = \int_0^T \int_{\mathcal{O}} c [\partial_t \psi_2 + \nabla \psi_2 \cdot u + \mu \Delta \psi_2] - nk(c) \psi_2 dx dt,$$

(3) For $e \in V$, $0 \leq t \leq T$,

$$\begin{aligned} \langle u(t), e \rangle_{H, H} &= \langle u_0, e \rangle_{H, H} - \nu \int_0^t \langle Au(s), e \rangle_{V^*, V} ds - \int_0^t \langle \mathcal{P}\{(u(s) \cdot \nabla)u(s)\}, e \rangle_{V^*, V} ds \\ &\quad - \int_0^t \langle \mathcal{P}\{n(s) \nabla \phi\}, e \rangle_{H, H} ds + \int_0^t \langle \sigma(u(s)) dW_s, e \rangle_{H, H} \end{aligned}$$

holds P -a.s..

Theorem 4.1 Suppose the assumptions (A)-(C) in Section 2 hold. Then, there exists a martingale weak solution to the stochastic Chemotaxis-Navier-Stokes system (1.1).

Proof. Let (n^m, c^m, u^m) be the solution constructed in Section 4.2. We will prove that the family $\{(n^m, c^m, u^m); m \geq 1\}$ is tight in the space $L^2([0, T], L^2(\mathcal{O})) \times L^2([0, T], H^1(\mathcal{O})) \times L^2([0, T], H)$. To this end, it suffices to show that the families $\{n^m; m \geq 1\}$, $\{c^m; m \geq 1\}$, $\{u^m; m \geq 1\}$ are respectively tight in the spaces $L^2([0, T], L^2(\mathcal{O}))$, $L^2([0, T], H^1(\mathcal{O}))$ and $L^2([0, T], H)$.

Define

$$\mathcal{Y} = \left\{ g \in L^2([0, T], H^2(\mathcal{O})), \quad \frac{dg}{dt} \in L^2([0, T], L^2(\mathcal{O})) \right\}$$

and the norm

$$\|g\|_{\mathcal{Y}} = \|g\|_{L^2([0, T], H^2(\mathcal{O}))} + \left\| \frac{dg}{dt} \right\|_{L^2([0, T], L^2(\mathcal{O}))}.$$

By Chapter III Theorem 2.1 in [16] (Page 271) and the Kondrachov embedding theorem, it is known that the embedding of \mathcal{Y} into $L^2([0, T], H^1(\mathcal{O}))$ is compact. By (4.30) and (4.31), we get that

$$\begin{aligned} P\left(\|c^m\|_{\mathcal{Y}}^2 \leq 2C_{\mu, T, N, C_M, \|c(0)\|_{H^1}}\right) \\ \geq P(\Omega_N^m) \\ \geq 1 - \frac{3C}{N}. \end{aligned}$$

Since we can choose the integer N as large as we wish, we conclude that the family $\{c^m\}$ is tight in $L^2([0, T], H^1(\mathcal{O}))$.

Similarly, (4.35) and (4.39) imply that $\{n^m\}$ is tight in $L^2([0, T], L^2(\mathcal{O}))$.

Given $\kappa \in (0, 1)$, let $W^{\kappa, 2}([0, T], V^*)$ be the Sobolev space of all $g \in L^2([0, T], V^*)$ such that

$$\int_0^T \int_0^T \frac{\|g(t) - g(s)\|_{V^*}^2}{|t - s|^{1+2\kappa}} dt ds < \infty,$$

endowed with the norm

$$\|g\|_{W^{\kappa, 2}([0, T], V^*)}^2 := \int_0^T \|g(t)\|_{V^*}^2 dt + \int_0^T \int_0^T \frac{\|g(t) - g(s)\|_{V^*}^2}{|t - s|^{1+2\kappa}} dt ds.$$

Set

$$B(u, u) := \mathcal{P}(u \cdot \nabla)u.$$

Recall

$$\langle B(u, u), v \rangle_{V^*, V} \leq C \|u\|_H \|u\|_V \|v\|_V, \quad u, v \in V.$$

It is known (see e.g. [18]) that B can be extended to a continuous operator

$$B : H \times H \rightarrow D(A^{-\varrho}) \quad (4.40)$$

for some $\varrho > 1$.

Using the equation satisfied by u^m , applying the similar arguments as in the proof of Theorem 3.1 in [6], we can show that

$$\mathbb{E}(\|u^m\|_{W^{\kappa, 2}([0, T], V^*)}) \leq C_\kappa. \quad (4.41)$$

Recall that the embedding of $L^2([0, T], V) \cap W^{\kappa, 2}([0, T], V^*)$ into $L^2([0, T], H)$ is compact (see e.g. Theorem 2.1 in [6]). (4.41) and (4.12) imply that $\{u^m\}$ is tight in $L^2([0, T], H)$.

On the other hand, by (d) and (e) in Corollary 4.1, applying Theorem 1 in [1] and Corollary 5.2 in [8], as the proof of Lemma 4.5 in [26], we can prove that $\{u^m\}$ is tight in $D([0, T], D(A^{-\varrho}))$, here ϱ is the constant appeared in (4.40) and $D([0, T], D(A^{-\varrho}))$ denotes the space of right continuous functions with left limits from $[0, T]$ into $D(A^{-\varrho})$ equipped with the Skorokhod topology. Moreover, since u^m takes values in $C([0, T], H)$, Proposition 1.6 in [8] implies that $\{u^m\}$ is also tight in $C([0, T], D(A^{-\varrho}))$ equipped with the usual uniform topology.

Now we have proved that $\{(n^m, c^m, u^m)\}$ is tight in the space:

$$\Pi := L^2([0, T], L^2(\mathcal{O})) \times L^2([0, T], H^1(\mathcal{O})) \times C([0, T], D(A^{-\varrho})) \cap L^2([0, T], H).$$

By the Skorokhod embedding theorem, there exist a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \in [0, T]}, \tilde{\mathbb{P}})$ and, on this basis, Π -valued random variables $(\tilde{n}^m, \tilde{c}^m, \tilde{u}^m)$, $(\tilde{n}, \tilde{c}, \tilde{u})$ such that

(I) $(\tilde{n}^m, \tilde{c}^m, \tilde{u}^m)$ has the same law as (n^m, c^m, u^m) ,

(II) $(\tilde{n}^m, \tilde{c}^m, \tilde{u}^m) \rightarrow (\tilde{n}, \tilde{c}, \tilde{u})$ in Π , $\tilde{\mathbb{P}}$ -a.s.

By a similar argument as in the proof of Theorem 3.1 in [6], we can show that

$$\tilde{u}(\cdot, \tilde{\omega}) \in C([0, T], D(A^{-\varrho})) \cap L^\infty([0, T], H) \cap L^2([0, T], V), \quad \tilde{\mathbb{P}}\text{-a.s.},$$

and there exists a U -cylindrical Wiener process \tilde{W} on the stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \in [0, T]}, \tilde{\mathbb{P}})$ such that $\tilde{\mathbb{P}}$ -a.s., the identity

$$\begin{aligned} \langle \tilde{u}(t), e \rangle_{H, H} &= \langle u_0, e \rangle_{H, H} - \nu \int_0^t \langle A \tilde{u}(s), e \rangle_{V^*, V} ds - \int_0^t \langle \mathcal{P}\{(\tilde{u}(s) \cdot \nabla) \tilde{u}(s)\}, e \rangle_{V^*, V} ds \\ &\quad - \int_0^t \langle \mathcal{P}\{\tilde{n}(s) \nabla \phi\}, e \rangle_{H, H} ds + \int_0^t \langle \sigma(\tilde{u}(s)) d\tilde{W}_s, e \rangle_{H, H} \end{aligned} \quad (4.42)$$

holds for all $t \in [0, T]$ and all $e \in D(A^e)$. Furthermore, it follows from (4.42) that $\tilde{u}(\cdot) \in C([0, T], H)$ $\tilde{\mathbb{P}}$ -a.s.. This can be seen as follows. Let $\tilde{L}(t)$ be the solution of the stochastic evolution equation:

$$d\tilde{L}(t) = A\tilde{L}(t)dt + \sigma(\tilde{u}(t))d\tilde{W}_t,$$

Then it is well known (see e.g. [13]) that $\tilde{L}(\cdot, \tilde{\omega}) \in C([0, T], H) \cap L^2([0, T], V)$ $\tilde{\mathbb{P}}$ -a.s.. On the other hand, using classical PDE arguments, Theorem 3.1 and Theorem 3.2 in [16], we can show that there exists a unique process $\tilde{Z} \in C([0, T], H) \cap L^2([0, T], V)$ satisfying the random PDE:

$$\begin{aligned} d\tilde{Z}(t) + \left((\tilde{Z}(t) + \tilde{L}(t)) \cdot \nabla \right) (\tilde{Z}(t) + \tilde{L}(t)) &= \nu \Delta \tilde{Z}(t) ds - \tilde{n}(t) \nabla \phi \\ \tilde{Z}(0) &= u_0. \end{aligned} \quad (4.43)$$

From the equation (4.42), it is easy to see that $\tilde{u} = \tilde{L} + \tilde{Z}$. Hence

$$\tilde{u}(\cdot, \tilde{\omega}) \in C([0, T], H) \cap L^2([0, T], V), \quad \tilde{\mathbb{P}}\text{-a.s.} \quad (4.44)$$

By a density argument, it is easy to see that the identity (4.42) holds for all $e \in V$.

Using the equations satisfied by (n^m, c^m, u^m) , we see that, for all $\psi_1, \psi_2 \in C^\infty([0, T] \times \mathcal{O})$ with compact supports with respect to the space variable, and $\psi_1(T, \cdot) = \psi_2(T, \cdot) = 0$,

$$\begin{aligned} \int_{\mathcal{O}} \psi_1(0, x) n_0^m dx &= \int_0^T \int_{\mathcal{O}} \tilde{n}^m [\partial_t \psi_1 + \nabla \psi_1 \cdot \tilde{u}^m + \delta \Delta \psi_1 + \nabla \psi_1 \cdot (\chi(\tilde{c}^m) \nabla \tilde{c}^m)] dx dt, \\ \int_{\mathcal{O}} \psi_2(0, x) c_0^m dx &= \int_0^T \int_{\mathcal{O}} \tilde{c}^m [\partial_t \psi_2 + \nabla \psi_2 \cdot \tilde{u}^m + \mu \Delta \psi_2] - \tilde{n}^m k(\tilde{c}^m) \psi_2 dx dt. \end{aligned}$$

Taking m into ∞ in the above two equations, we see that $(\tilde{n}, \tilde{c}, \tilde{u})$ satisfies (2) in Definition 4.1.

Finally, by (a) (b) (c) (d) in Corollary 4.1, (I) (II), (4.30) and (4.44), we see that $(\tilde{n}, \tilde{c}, \tilde{u})$ satisfies (1) in Definition 4.1. Hence, $(\tilde{n}, \tilde{c}, \tilde{u})$ is a martingale weak solution. \square

4.4 Pathwise Weak Solution

Theorem 4.2 *Assume, in addition, that the function $\chi(\cdot)$ is a positive constant. Then there exists a unique pathwise weak solution to the stochastic Chemotaxis-Navier-Stokes system (1.1).*

Proof. From Theorem 4.1, we already know that there exists a martingale weak solution to system (1.1). By the Watanabe and Yamada Theorem, we will complete the proof of the theorem if we can show the pathwise uniqueness of the solutions. That is what we will do in the remaining part of the proof. Without loss of generality, we assume $\chi(\cdot) \equiv 1$.

Assume that (n_1, c_1, u_1) and (n_2, c_2, u_2) are two solutions of the system (1.1) on the same probability basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$, with a same U -valued cylindrical Wiener process W . We will prove that

$$(n_1, c_1, u_1) = (n_2, c_2, u_2).$$

For simplicity, set

$$n^\Delta = n_1 - n_2, \quad c^\Delta = c_1 - c_2, \quad u^\Delta = u_1 - u_2.$$

By chain rule,

$$\begin{aligned} &\|n^\Delta(t)\|_{L^2}^2 + 2\delta \int_0^t \|\nabla n^\Delta(s)\|_{L^2}^2 ds \\ &= 2 \int_0^t \langle u_1(s) n_1(s) - u_2(s) n_2(s), \nabla n^\Delta(s) \rangle_{L^2, L^2} ds \\ &\quad + 2 \int_0^t \langle n_1(s) \nabla c_1(s) - n_2(s) \nabla c_2(s), \nabla n^\Delta(s) \rangle_{L^2, L^2} ds \\ &\leq \delta \int_0^t \|\nabla n^\Delta(s)\|_{L^2}^2 ds + \frac{2}{\delta} \int_0^t \|u_1(s) n_1(s) - u_2(s) n_2(s)\|_{L^2}^2 ds \\ &\quad + \frac{2}{\delta} \int_0^t \|n_1(s) \nabla c_1(s) - n_2(s) \nabla c_2(s)\|_{L^2}^2 ds \\ &= \delta \int_0^t \|\nabla n^\Delta(s)\|_{L^2}^2 ds + I_1^n(t) + I_2^n(t). \end{aligned} \quad (4.45)$$

By (4.24) and (4.25), for $I_1^n(t)$, we have for $\varepsilon > 0$,

$$\begin{aligned}
I_1^n(t) &\leq C_\delta \left(\int_0^t \|u_1(s)\|_{L^4}^2 \|n^\Delta(s)\|_{L^2}^2 ds + \int_0^t \|u^\Delta(s)\|_{L^4}^2 \|n_2(s)\|_{L^4}^2 ds \right) \\
&\leq C_\delta \left(\int_0^t \|u_1(s)\|_{L^2} \|\nabla u_1(s)\|_{L^2} \left(\|\nabla n^\Delta(s)\|_{L^2} \|n^\Delta(s)\|_{L^2} \right. \right. \\
&\quad \left. \left. + \|n^\Delta(s)\|_{L^2}^2 \right) ds \right. \\
&\quad \left. + \int_0^t \|u^\Delta(s)\|_{L^2} \|\nabla u^\Delta(s)\|_{L^2} \left(\|n_2(s)\|_{L^2} \|\nabla n_2(s)\|_{L^2} + \|n_2(s)\|_{L^2}^2 \right) ds \right) \\
&\leq \frac{\delta}{4} \int_0^t \|\nabla n^\Delta(s)\|_{L^2}^2 ds \\
&\quad + C_\delta \int_0^t \|n^\Delta(s)\|_{L^2}^2 \left(\|u_1(s)\|_{L^2}^2 \|\nabla u_1(s)\|_{L^2}^2 + \|u_1(s)\|_{L^2} \|\nabla u_1(s)\|_{L^2} \right) ds \\
&\quad + \epsilon \int_0^t \|\nabla(u^\Delta(s))\|_{L^2}^2 ds \\
&\quad + C_{\delta,\epsilon} \int_0^t \|u^\Delta(s)\|_{L^2}^2 \left(\|n_2(s)\|_{L^2}^2 \|\nabla n_2(s)\|_{L^2}^2 + \|n_2(s)\|_{L^2}^4 \right) ds.
\end{aligned} \tag{4.46}$$

By (4.25) and (4.26), for $\varepsilon > 0$ we have

$$\begin{aligned}
I_2^n(t) &\leq C_\delta \left(\int_0^t \|n_1(s) \nabla c^\Delta(s)\|_{L^2}^2 ds + \int_0^t \|n^\Delta(s) \nabla c_2(s)\|_{L^2}^2 ds \right) \\
&\leq C_\delta \left(\int_0^t \|n_1(s)\|_{L^4}^2 \|\nabla c^\Delta(s)\|_{L^4}^2 ds + \int_0^t \|n^\Delta(s)\|_{L^4}^2 \|\nabla c_2(s)\|_{L^4}^2 ds \right) \\
&\leq C_\delta \int_0^t \left(\|n_1(s)\|_{L^2} \|\nabla n_1(s)\|_{L^2} + \|n_1(s)\|_{L^2}^2 \right) \\
&\quad \cdot \left(\|c^\Delta(s)\|_{H^2} \|\nabla c^\Delta(s)\|_{L^2} + \|\nabla c^\Delta(s)\|_{L^2}^2 \right) ds \\
&\quad + C_\delta \int_0^t \left(\|n^\Delta(s)\|_{L^2} \|\nabla n^\Delta(s)\|_{L^2} + \|n^\Delta(s)\|_{L^2}^2 \right) \\
&\quad \cdot \left(\|c_2(s)\|_{H^2} \|\nabla c_2(s)\|_{L^2} + \|\nabla c_2(s)\|_{L^2}^2 \right) ds \\
&\leq \epsilon \int_0^t \|c^\Delta(s)\|_{H^2}^2 ds \\
&\quad + C_{\delta,\epsilon} \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 \\
&\quad \cdot \left(\|n_1(s)\|_{L^2}^2 \|\nabla n_1(s)\|_{L^2}^2 + \|n_1(s)\|_{L^2}^4 + \|n_1(s)\|_{L^2} \|\nabla n_1(s)\|_{L^2} + \|n_1(s)\|_{L^2}^2 \right) ds \\
&\quad + \frac{\delta}{4} \int_0^t \|\nabla n^\Delta(s)\|_{L^2}^2 ds \\
&\quad + C_\delta \int_0^t \|n^\Delta(s)\|_{L^2}^2 \\
&\quad \cdot \left(\|c_2(s)\|_{H^2}^2 \|\nabla c_2(s)\|_{L^2}^2 + \|\nabla c_2(s)\|_{L^2}^4 + \|c_2(s)\|_{H^2} \|\nabla c_2(s)\|_{L^2} + \|\nabla c_2(s)\|_{L^2}^2 \right) ds.
\end{aligned} \tag{4.47}$$

Combining (4.45)–(4.47), we have

$$\begin{aligned}
& \|n^\Delta(t)\|_{L^2}^2 + \frac{\delta}{2} \int_0^t \|\nabla n^\Delta(s)\|_{L^2}^2 ds \\
\leq & C_\delta \int_0^t \|n^\Delta(s)\|_{L^2}^2 \left(\|u_1(s)\|_{L^2}^2 \|\nabla u_1(s)\|_{L^2}^2 + \|u_1(s)\|_{L^2} \|\nabla u_1(s)\|_{L^2} \right) ds \\
& + \epsilon \int_0^t \|\nabla(u^\Delta(s))\|_{L^2}^2 ds \\
& + C_{\delta,\epsilon} \int_0^t \|u^\Delta(s)\|_{L^2}^2 \left(\|n_2(s)\|_{L^2}^2 \|\nabla n_2(s)\|_{L^2}^2 + \|n_2(s)\|_{L^2}^4 \right) ds \\
& + \epsilon \int_0^t \|c^\Delta(s)\|_{H^2}^2 ds \\
& + C_{\delta,\epsilon} \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 \\
& \quad \cdot \left(\|n_1(s)\|_{L^2}^2 \|\nabla n_1(s)\|_{L^2}^2 + \|n_1(s)\|_{L^2}^4 + \|n_1(s)\|_{L^2} \|\nabla n_1(s)\|_{L^2} + \|n_1(s)\|_{L^2}^2 \right) ds \\
& + C_\delta \int_0^t \|n^\Delta(s)\|_{L^2}^2 \\
& \quad \cdot \left(\|c_2(s)\|_{H^2}^2 \|\nabla c_2(s)\|_{L^2}^2 + \|\nabla c_2(s)\|_{L^2}^4 + \|c_2(s)\|_{H^2} \|\nabla c_2(s)\|_{L^2} + \|\nabla c_2(s)\|_{L^2}^2 \right) ds.
\end{aligned} \tag{4.48}$$

Now we estimate $\|c^\Delta\|_{H^1}^2$. By the chain rule, we have

$$\begin{aligned}
& \|c^\Delta(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 ds \\
\leq & 2 \int_0^t \langle u_1(s)c_1(s) - u_2(s)c_2(s), \nabla c^\Delta(s) \rangle_{L^2, L^2} ds \\
& - 2 \int_0^t \langle k(c_1(s))n_1(s) - k(c_2(s))n_2(s), c^\Delta(s) \rangle_{L^2, L^2} ds \\
\leq & \frac{\mu}{2} \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 ds + \frac{2}{\mu} \int_0^t \|u_1(s)c_1(s) - u_2(s)c_2(s)\|_{L^2}^2 ds \\
& + \int_0^t \|k(c_1(s))n_1(s) - k(c_2(s))n_2(s)\|_{L^2}^2 ds + \int_0^t \|c^\Delta(s)\|_{L^2}^2 ds,
\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}
& \|\nabla c^\Delta(t)\|_{L^2}^2 + 2\mu \int_0^t \|\Delta c^\Delta(s)\|_{L^2}^2 ds \\
= & -2 \int_0^t \langle u_1(s) \cdot \nabla c_1(s) - u_2(s) \cdot \nabla c_2(s), \Delta c^\Delta(s) \rangle_{L^2, L^2} ds \\
& - 2 \int_0^t \langle k(c_1(s))n_1(s) - k(c_2(s))n_2(s), \Delta c^\Delta(s) \rangle_{L^2, L^2} ds \\
\leq & \mu \int_0^t \|\Delta c^\Delta(s)\|_{L^2}^2 ds + \frac{2}{\mu} \int_0^t \|u_1(s) \cdot \nabla c_1(s) - u_2(s) \cdot \nabla c_2(s)\|_{L^2}^2 ds \\
& + \frac{2}{\mu} \int_0^t \|k(c_1(s))n_1(s) - k(c_2(s))n_2(s)\|_{L^2}^2 ds.
\end{aligned} \tag{4.50}$$

By the similar arguments as in the proof of (4.46), we have

$$\begin{aligned}
& \frac{2}{\mu} \int_0^t \|u_1(s)c_1(s) - u_2(s)c_2(s)\|_{L^2}^2 ds \\
& \leq \frac{\mu}{2} \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 ds \\
& \quad + C_\mu \int_0^t \|c^\Delta(s)\|_{L^2}^2 \left(\|u_1(s)\|_{L^2}^2 \|\nabla u_1(s)\|_{L^2}^2 + \|u_1(s)\|_{L^2} \|\nabla u_1(s)\|_{L^2} \right) ds \\
& \quad + \epsilon \int_0^t \|\nabla u^\Delta(s)\|_{L^2}^2 ds \\
& \quad + C_{\mu, \epsilon} \int_0^t \|u^\Delta(s)\|_{L^2}^2 \left(\|c_2(s)\|_{L^2}^2 \|\nabla c_2(s)\|_{L^2}^2 + \|c_2(s)\|_{L^2}^4 \right) ds.
\end{aligned} \tag{4.51}$$

Furthermore, for $\varepsilon > 0$,

$$\begin{aligned}
& \int_0^t \|k(c_1(s))n_1(s) - k(c_2(s))n_2(s)\|_{L^2}^2 ds \\
& \leq 2 \left(\int_0^t \|(k(c_1(s)) - k(c_2(s)))n_1(s)\|_{L^2}^2 ds + \int_0^t \|k(c_2(s))n^\Delta(s)\|_{L^2}^2 ds \right) \\
& \leq \sup_{r \in [0, C_M]} |k'(r)|^2 \int_0^t \|c^\Delta(s)\|_{L^2}^2 \|n_1(s)\|_{L^2}^2 ds + \sup_{r \in [0, C_M]} |k(r)|^2 \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\
& \leq C \int_0^t \|c^\Delta(s)\|_{L^4}^2 \|n_1(s)\|_{L^4}^2 ds + C \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\
& \leq C \int_0^t \left(\|\nabla c^\Delta(s)\|_{L^2} \|c^\Delta(s)\|_{L^2} + \|c^\Delta(s)\|_{L^2}^2 \right) \left(\|\nabla n_1(s)\|_{L^2} \|n_1(s)\|_{L^2} + \|n_1(s)\|_{L^2}^2 \right) ds \\
& \quad + C \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\
& \leq \epsilon \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 ds + C \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\
& \quad + C_\epsilon \int_0^t \|c^\Delta(s)\|_{L^2}^2 \left(\|\nabla n_1(s)\|_{L^2}^2 \|n_1(s)\|_{L^2}^2 + \|n_1(s)\|_{L^2}^4 + \|\nabla n_1(s)\|_{L^2} \|n_1(s)\|_{L^2} + \|n_1(s)\|_{L^2}^2 \right) ds,
\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
& \frac{2}{\mu} \int_0^t \|u_1(s) \cdot \nabla c_1(s) - u_2(s) \cdot \nabla c_2(s)\|_{L^2}^2 ds \\
& \leq C_\mu \int_0^t \|u^\Delta(s) \cdot \nabla c_1(s)\|_{L^2}^2 ds + C_\mu \int_0^t \|u_2(s) \cdot \nabla c^\Delta(s)\|_{L^2}^2 ds \\
& \leq C_\mu \int_0^t \|u^\Delta(s)\|_{L^4}^2 \|\nabla c_1(s)\|_{L^4}^2 ds + C_\mu \int_0^t \|u_2(s)\|_{L^4}^2 \|\nabla c^\Delta(s)\|_{L^4}^2 ds \\
& \leq C_\mu \int_0^t \|u^\Delta(s)\|_{L^2} \|\nabla u^\Delta(s)\|_{L^2} \left(\|c_1(s)\|_{H^2} \|\nabla c_1(s)\|_{L^2} + \|\nabla c_1(s)\|_{L^2}^2 \right) ds \\
& \quad + C_\mu \int_0^t \|u_2(s)\|_{L^2} \|\nabla u_2(s)\|_{L^2} \left(\|c^\Delta(s)\|_{H^2} \|\nabla c^\Delta(s)\|_{L^2} + \|\nabla c^\Delta(s)\|_{L^2}^2 \right) ds \\
& \leq \epsilon \int_0^t \|\nabla u^\Delta(s)\|_{L^2}^2 ds \\
& \quad + C_{\epsilon, \mu} \int_0^t \|u^\Delta(s)\|_{L^2}^2 \left(\|c_1(s)\|_{H^2}^2 \|\nabla c_1(s)\|_{L^2}^2 + \|\nabla c_1(s)\|_{L^2}^4 \right) ds \\
& \quad + \epsilon \int_0^t \|c^\Delta(s)\|_{H^2}^2 ds \\
& \quad + C_{\epsilon, \mu} \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 \left(\|u_2(s)\|_{L^2}^2 \|\nabla u_2(s)\|_{L^2}^2 + \|u_2(s)\|_{L^2} \|\nabla u_2(s)\|_{L^2} \right) ds.
\end{aligned} \tag{4.53}$$

By (4.27), we have

$$\epsilon \int_0^t \|c^\Delta(s)\|_{H^2}^2 ds \leq \epsilon C \left(\int_0^t \|c^\Delta(s)\|_{L^2}^2 ds + \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 ds + \int_0^t \|\Delta c^\Delta(s)\|_{L^2}^2 ds \right). \quad (4.54)$$

Combining (4.49)–(4.54), we arrive at

$$\begin{aligned} & \|c^\Delta(t)\|_{L^2}^2 + \|\nabla c^\Delta(t)\|_{L^2}^2 + (\mu - \epsilon(1 + \frac{2}{\mu}) - \epsilon C) \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 ds + (\mu - \epsilon C) \int_0^t \|\Delta c^\Delta(s)\|_{L^2}^2 ds \\ \leq & 2\epsilon \int_0^t \|\nabla u^\Delta(s)\|_{L^2}^2 ds + C_\mu \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\ & + C_{\mu,\epsilon} \int_0^t \|c^\Delta(s)\|_{L^2}^2 \left(1 + \|u_1(s)\|_{L^2}^2 \|\nabla u_1(s)\|_{L^2}^2 + \|u_1(s)\|_{L^2} \|\nabla u_1(s)\|_{L^2} \right. \\ & \quad \left. + \|n_1(s)\|_{L^2}^2 \|\nabla n_1(s)\|_{L^2}^2 + \|n_1(s)\|_{L^2}^4 \right. \\ & \quad \left. + \|n_1(s)\|_{L^2} \|\nabla n_1(s)\|_{L^2} + \|n_1(s)\|_{L^2}^2 \right) ds \\ & + C_{\epsilon,\mu} \int_0^t \|u^\Delta(s)\|_{L^2}^2 \left(\sum_{i=1}^2 \left(\|c_i(s)\|_{L^2}^2 \|\nabla c_i(s)\|_{L^2}^2 + \|c_i(s)\|_{L^2}^4 \right) \right) ds \\ & + C_{\epsilon,\mu} \int_0^t \|\nabla c^\Delta(s)\|_{L^2}^2 \left(\|u_2(s)\|_{L^2}^2 \|\nabla u_2(s)\|_{L^2}^2 + \|u_2(s)\|_{L^2} \|\nabla u_2(s)\|_{L^2} \right) ds. \end{aligned} \quad (4.55)$$

Since

$$2 \left| \langle \mathcal{P}\{(u_1(t) \cdot \nabla)u_1(t) - (u_2(t) \cdot \nabla)u_2(t)\}, u^\Delta(t) \rangle_{V',V} \right| \leq \nu \|\nabla u^\Delta(t)\|_{L^2}^2 + C_\nu \|u_2(t)\|_{L^4}^4 \|u^\Delta(t)\|_{L^2}^2,$$

by Itô's formula, we have

$$\begin{aligned} & \|u^\Delta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u^\Delta(s)\|_{L^2}^2 ds \\ = & -2 \int_0^t \langle \mathcal{P}\{(u_1(s) \cdot \nabla)u_1(s) - (u_2(s) \cdot \nabla)u_2(s)\}, u^\Delta(s) \rangle_{V',V} ds \\ & -2 \int_0^t \langle \mathcal{P}\{n_1(s)\nabla\phi - n_2(s)\nabla\phi\}, u^\Delta(s) \rangle_{L^2,L^2} ds \\ & + 2 \int_0^t \langle \sigma(u_1(s)) - \sigma(u_2(s)), u^\Delta(s) \rangle_{L^2,L^2} dW(s) + 2 \int_0^t \|\sigma(u_1(s)) - \sigma(u_2(s))\|_{L_0^2}^2 ds \\ \leq & \nu \int_0^t \|\nabla u^\Delta(s)\|_{L^2}^2 ds + C_\nu \int_0^t \|u_2(s)\|_{L^4}^4 \|u^\Delta(s)\|_{L^2}^2 ds \\ & + \int_0^t \|u^\Delta(s)\|_{L^2}^2 ds + \|\nabla\phi\|_\infty^2 \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\ & + 2 \int_0^t \langle \sigma(u_1(s)) - \sigma(u_2(s)), u^\Delta(s) \rangle_{L^2,L^2} dW(s) + C \int_0^t \|u^\Delta(s)\|_{L^2}^2 ds \\ \leq & \nu \int_0^t \|\nabla u^\Delta(s)\|_{L^2}^2 ds + C_\nu \int_0^t \|u_2(s)\|_{L^2}^2 \|\nabla u_2(s)\|_{L^2}^2 \|u^\Delta(s)\|_{L^2}^2 ds \\ & + C \int_0^t \|u^\Delta(s)\|_{L^2}^2 ds + C_\phi \int_0^t \|n^\Delta(s)\|_{L^2}^2 ds \\ & + 2 \int_0^t \langle \sigma(u_1(s)) - \sigma(u_2(s)), u^\Delta(s) \rangle_{L^2,L^2} dW(s). \end{aligned} \quad (4.56)$$

Set

$$\Lambda(s) := \|n^\Delta(s)\|_{L^2}^2 + \|c^\Delta(s)\|_{L^2}^2 + \|\nabla c^\Delta(s)\|_{L^2}^2 + \|u^\Delta(s)\|_{L^2}^2.$$

Choosing ϵ sufficiently small, by (4.48)(4.54) (4.55) and (4.56), we get that

$$\Lambda(t) \leq C_{\epsilon,\mu,\nu,\delta} \int_0^t \Lambda(s) \Xi(s) ds + 2 \int_0^t \langle \sigma(u_1(s)) - \sigma(u_2(s)), u^\Delta(s) \rangle_{L^2,L^2} dW_s. \quad (4.57)$$

here

$$\begin{aligned}\Xi(s) = & 1 + \sum_{i=1}^2 \sum_{j=1}^2 \left(\|n_i(s)\|_{L^2}^j \|\nabla n_i(s)\|_{L^2}^j + \|n_i(s)\|_{L^2}^{2j} + \|c_i(s)\|_{L^2}^j \|\nabla c_i(s)\|_{L^2}^j + \|c_i(s)\|_{L^2}^{2j} \right. \\ & \left. + \|c_i(s)\|_{H^2}^j \|\nabla c_i(s)\|_{L^2}^j + \|\nabla c_i(s)\|_{L^2}^{2j} + \|u_i(s)\|_{L^2}^j \|\nabla u_i(s)\|_{L^2}^j \right).\end{aligned}$$

By the Gronwall' lemma, we arrive at

$$\Lambda(t) \leq 2 \sup_{s \in [0, t]} \left| \int_0^s \langle \sigma(u_1(s)) - \sigma(u_2(s)), u^\Delta(s) \rangle_{L^2, L^2} dW(s) \right| \cdot e^{C_{\epsilon, \mu, \nu, \delta} \int_0^t \Xi(s) ds} \quad (4.58)$$

Define

$$\begin{aligned}\tau_i^N = & \inf_{t \geq 0} \{ \sup_{s \in [0, t]} \|n_i(s)\|_{L^2}^2 \vee \int_0^t \|\nabla n_i(s)\|_{L^2}^2 ds \vee \sup_{s \in [0, t]} \|c_i(s)\|_{H^1}^2 \vee \int_0^t \|c_i(s)\|_{H^2}^2 ds \\ & \vee \sup_{s \in [0, t]} \|u_i(s)\|_{L^2}^2 \vee \int_0^t \|\nabla u_i(s)\|_{L^2}^2 ds \geq N \} \wedge T, \quad i = 1, 2.\end{aligned}$$

Put $\tau_N = \tau_1^N \wedge \tau_2^N$. Because (n_i, c_i, u_i) , $i = 1, 2$ satisfy (1) in Definition 4.1, we see that

$$\sup_{s \in [0, T]} \|c_i(s)\|_{H^1}^2 + \int_0^T \|c_i(s)\|_{H^2}^2 ds + \sup_{s \in [0, T]} \|u_i(s)\|_{L^2}^2 + \int_0^T \|\nabla u_i(s)\|_{L^2}^2 ds < \infty, \quad \mathbb{P}\text{-a.s.} \quad (4.59)$$

Repeating the arguments in Subsection 4.2, we can get the following result (see Remark 4.1):

$$\sup_{s \in [0, T]} \|n_i(s)\|_{L^2}^2 + \int_0^T \|\nabla n_i(s)\|_{L^2}^2 ds < \infty, \quad \mathbb{P}\text{-a.s.} \quad (4.60)$$

(4.59) and (4.60) imply that

$$\tau_N \nearrow T, \quad \mathbb{P}\text{-a.s.} \quad (4.61)$$

Replace t by $t \wedge \tau_N$ in (4.58) to get

$$\begin{aligned}\Lambda(t \wedge \tau_N) & \leq 2 \sup_{s \in [0, t \wedge \tau_N]} \left| \int_0^s \langle (\sigma(u_1(s)) - \sigma(u_2(s))) dW_s, u^\Delta(s) \rangle_{L^2, L^2} \right| \cdot e^{C_{\epsilon, \mu, \nu, \delta} \int_0^{t \wedge \tau_N} \Xi(s) ds} \\ & \leq C_{\epsilon, \mu, \nu, \delta, N} \sup_{s \in [0, t]} \left| \int_0^{s \wedge \tau_N} \langle (\sigma(u_1(s)) - \sigma(u_2(s))) dW_s, u^\Delta(s) \rangle_{L^2, L^2} \right|.\end{aligned} \quad (4.62)$$

By BDG inequality,

$$\begin{aligned}\mathbb{E} \left(\sup_{t \in [0, T]} \Lambda(t \wedge \tau_N) \right) & \leq C_{\epsilon, \mu, \nu, \delta, N} \mathbb{E} \left(\sup_{s \in [0, T]} \left| \int_0^{s \wedge \tau_N} \langle (\sigma(u_1(s)) - \sigma(u_2(s))) dW_s, u^\Delta(s) \rangle_{L^2, L^2} \right| \right) \\ & \leq C_{\epsilon, \mu, \nu, \delta, N} \mathbb{E} \left(\left| \int_0^{T \wedge \tau_N} \|u^\Delta(s)\|_{L^2}^4 ds \right|^{1/2} \right) \\ & \leq \frac{1}{2} \mathbb{E} \left(\sup_{t \in [0, T]} \|u^\Delta(t \wedge \tau_N)\|_{L^2}^2 \right) + C_{\epsilon, \mu, \nu, \delta, N} \mathbb{E} \left(\int_0^{T \wedge \tau_N} \|u^\Delta(s)\|_{L^2}^2 ds \right) \\ & \leq \frac{1}{2} \mathbb{E} \left(\sup_{t \in [0, T]} \|u^\Delta(t \wedge \tau_N)\|_{L^2}^2 \right) + C_{\epsilon, \mu, \nu, \delta, N} \int_0^T \mathbb{E} \left(\sup_{s \in [0, t]} \|u^\Delta(s \wedge \tau_N)\|_{L^2}^2 dt \right).\end{aligned}$$

By the Gronwall's lemma, we obtain

$$\mathbb{E} \left(\sup_{t \in [0, T]} \Lambda(t \wedge \tau_N) \right) = 0.$$

Let $N \rightarrow \infty$ to obtain

$$\mathbb{E} \left(\sup_{t \in [0, T]} \Lambda(t) \right) = 0,$$

which implies the uniqueness. \square

Acknowledgements This work is partly supported by National Natural Science Foundation of China (No.11671372, No.11431014, No.11401557), the Fundamental Research Funds for the Central Universities (No. WK 3470000008), and Key Research Program of Frontier Sciences CAS(No. QYZDB-SSW-SYS009).

References

- [1] D. Aldous, Stopping times and tightness, *Ann. Probab.* 6(1978), 335-340.
- [2] X., Cao, Global classical solutions in chemotaxis(-Navier)-Stokes system with rotational flux term. *J. Differential Equations* 261, (2016), no. 12, 6883-6914.
- [3] M., Chae, K., Kang, J., Lee, Global existence and temporal decay in Keller-Segel models coupled to fluid equations. *Comm. Partial Differential Equations* 39, (2014), no. 7, 1205-1235.
- [4] R. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, *Comm. Partial Differential Equations* 35, (2010) 1635-1673. MR2754058
- [5] R., Duan, Z., Xiang, A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion. *Int. Math. Res. Not. IMRN* 2014, no. 7, 1833-1852.
- [6] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields* 102, (1995), 367-391.
- [7] S., Ishida, Global existence and boundedness for chemotaxis-Navier-Stokes systems with position-dependent sensitivity in 2D bounded domains. *Discrete Contin. Dyn. Syst.* 35, (2015), no. 8, 3463-3482.
- [8] A. Jakubowski, On the Skorokhod topology, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, 22(1986), 263-285.
- [9] H., Kozono, M., Miura, Y., Sugiyama, Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid. *J. Funct. Anal.* 270, (2016), no. 5, 1663-1683.
- [10] Y., Li, Y.X., Li, Global boundedness of solutions for the chemotaxis-Navier-Stokes system in R^2 . *J. Differential Equations* 261, (2016) no. 11, 6570-6613.
- [11] J. Liu, A. Lorz, A coupled chemotaxis-fluid model: Global existence. *Ann. I.H. Poincaré-AN* 28, (2011), 643-652.
- [12] A. Lorz, Coupled chemotaxis fluid model, *Math. Models Methods Appl. Sci.* 20, (2010) 987C1004. MR2659745
- [13] C. Prévôt and M. Röckner, A concise course on stochastic partial differential equations, SpringerVerlag, 2007.
- [14] Y., Tao, M., Winkler, Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion. *Ann. Inst. H. Poincare Anal. Non Lineaire* 30, (2013), no. 1, 157-178.
- [15] Michael E. Taylor, Partial Differential Equations I: Basic Theory. 2en Edition, Applied Mathematical Sciences, 115, Springer
- [16] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis., Revised ed. (North-Holland Publishing Co., Amsterdam, 1979). MR 0603444
- [17] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, *Proc. Natl. Acad. Sci. USA* 102, (2005) 2277-2282.
- [18] M.I., Visik, A. V., Fursikov, *Mathematical problems of statistical Hydromechanics*, Klurer, Dordrecht, 1980.
- [19] M., Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity. *Calc. Var. Partial Differential Equations* 54, (2015), no. 4, 3789-3828.
- [20] M. Winkler, Global Large-Data Solutions in a Chemotaxis-(Navier-)Stokes System Modeling Cellular Swimming in Fluid Drops. *Comm. Part. Diff. Eqs.* 37, (2012), 319-351.
- [21] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. *J. Differential Equations* 248, (2010), 2889-2905.
- [22] M., Winkler, Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system. *Ann. Inst. H. Poincare Anal. Non Lineaire* 33, (2016), no. 5, 1329-1352.
- [23] M., Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities. *SIAM J. Math. Anal.* 47, (2015), no. 4, 3092-3115.

- [24] M., Winkler, Michael, Stabilization in a two-dimensional chemotaxis-Navier-Stokes system. *Arch. Ration. Mech. Anal.* 211, (2014), no. 2, 455-487.
- [25] C. Xue, H.G. Othmer, Multiscale models of taxis-driven patterning in bacterial populations, *SIAM J. Appl. Math.* 70, (1) (2009) 133-167. MR2505083
- [26] J.L., Zhai, T.S., Zhang, Large deviations for 2-D stochastic Navier-Stokes equations driven by multiplicative Lévy noises. *Bernoulli* 21, (2015), 2351-2392.
- [27] Q., Zhang, X., Zheng, Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations. *SIAM J. Math. Anal.* 46, (2014), no. 4, 3078-3105.